The Dynamics of A Cylinder Approaching a Wall in a Viscous Fluid

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As a cylinder falls in close proximity with a wall in a viscous fluid, the flow in the gap between them is dominated by the viscous effect, and can be approximated by the lubrication theory. Here we analyze and compute the detailed dynamic behaviors of the cylinder, and discuss their dependence on the controlling parameters. We contrast the dynamics of a cylinder and a sphere, and also compare the cases with and without gravity. We show that without gravity, the cylinder comes to rest asymptotically at a finite separation from the wall, and contact can never happen; and with gravity, the cylinder comes to rest and collides with the wall asymptotically, and contact can not happen in a finite time. We also show that the continuum limit for the cylinder holds for a larger range of the Stokes number and under matching conditions, a longer time than the sphere. Our results serve as a building block for integrating the lubrication theory with computations of Navier-Stokes equations to handle particle interactions in a flow.

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I. INTRODUCTION

The interaction of particles in fluids is key to understanding collective behavior of particles in particle-laden flows such as sedimentation[2,6], particle suspensions[14,16], cloud formation[8,15], as well as biological phenomena such as ocean bio-mixing and nutrient transport[12,13]. In the high particle-density limit, close-range particle interactions become an important feature of such flows. They introduce small length and time scales in the otherwise inertia-dominated flows.

One question about the interaction of two particles in a fluid is whether they bounce off each other or will they stick together? Unlike dry collisions, the dynamics of two particles approaching each other is dictated by lubrication force when the gap between them is small. The classical lubrication theory predicts that contact and rebound of two particles would not be possible because the hydrodynamic force diverges as the gap separation tends to zero[1,7]. A key parameter in studying particle collisions in fluids is the Stokes number. It is defined as the ratio of the particle inertia to the Stoke drag. More recent experimental and theoretical studies[3,4,5,10,11,20] have shown that there is no contact or rebound if the Stokes number is less than critical values, but contact and rebound can occur if the Stokes number is sufficiently high. The classical lubrication theory needs to be augmented to account for the effects of the compressibility and non-continuum of fluids[5] and the deformation and roughness of particles[3,4] to make contact and rebound possible at high Stokes numbers. When the Stokes number is low, the particle inertia is small relative to the viscous drag, and the kinetic energy of the particle can not compensate for the viscous dissipation. So particles approaching each other by inertia slow down and come to a rest at a given separation, and no contact or rebound is to take place.

In this paper, we apply the lubrication theory to study the dynamics of a cylinder approaching a wall in a viscous fluid with or without gravity. We show that without gravity, the cylinder comes to rest asymptotically at a finite separation from the wall, and contact can never happen; and with gravity, a constant force driving the cylinder toward the wall, the cylinder comes to rest and collides with the wall asymptotically, and contact can not happen in a finite time. These results for the cylinder in 2D are similar to a sphere in 3D. We present detailed dynamics of the cylinder and discuss how they are affected by controlling parameters and initial conditions. We also compare the dynamics of the cylinder with a sphere and show that the continuum limit for the cylinder holds for a larger range of the Stokes number and under matching conditions, a longer time than the sphere.

The diminishing gap between particles and the associated small time scale introduce numerical difficulties in simulating flows with interacting particles. A brute force method would be to refine the grid spacing and reduce the time step, but both would increase the computational cost significantly, as the gap size can become very small. Many computational schemes avoid this problem by introducing ad-hoc collision laws, such as a dry collision model without taking into account the fluid effect, which can lead to erroneous result. An alternative method is to integrate the lubrication approximation for flow inside the gap with the underlying numerical algorithm. Such a method would be accurate and efficient for simulating flows without rebound at relatively low Stokes numbers.

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This paper provides formulas and validation results for developing such a method.

II. MODELS

We consider the case of a cylinder falling vertically toward a fixed wall in an incompressible viscous fluid, as illustrated in figure ??.

The flow around the cylinder is governed by the Navier-Stokes equations

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\nabla p + \nu^* \nabla^2 \vec{v} \ , \tag{1a}$$

$$\nabla \cdot \vec{v} = 0 \ , \tag{1b}$$

where $\vec{v} = (u, v)$ is the fluid velocity, p is the fluid pressure, and ν^* is the nondimensional kinematic viscosity of the fluid. Hereafter, all variables and quantities are nondimensionalized with the cylinder diameter $\mathtt{D}$, the velocity scale $\mathtt{U} = \sqrt{\mathtt{gD}}$ (where $\mathtt{g}$ is the gravitational constant), and the fluid density $\rho_\mathtt{f}$ unless otherwise specified. So $\nu^* = \mu_\mathtt{f}/(\rho_\mathtt{f}\mathtt{UD})$, where $\mu_\mathtt{f}$ is the dimensional dynamic viscosity of the fluid.

When the cylinder is in close proximity to the wall, the flow in the gap between the cylinder and the wall can be approximated by the lubrication equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ , \tag{2a}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\nu^*} \frac{dp}{dx} \ . \tag{2b}$$

Equation 2b is valid when $h_m \ll 1$ and $h_m^2/\nu^* \ll 1$, where h_m is the minimum gap clearance in the wall-normal direction (nondimensionalized by the diameter of the cylinder $\mathtt{D}$). These are the two conditions for the lubrication theory to hold. At this lubrication limit, the flow in the gap can be considered as quasi-steady. Equations 2b and 1b can be integrated to give

$$u = \frac{1}{2\nu} \frac{dp}{dx} y(y-h) \ , \tag{3a}$$

$$\frac{dp}{dx} = 12\nu^* \frac{x}{h^3} \frac{\partial h}{\partial t} \ , \tag{3b}$$

where the height h at the abscissa x is illustrated in figure ??.

The dynamics of the cylinder settling under gravity are governed by the following ODE

$$m_s \frac{dv_c}{dt} = -(m_s - m_f) g^* + F_f \ , \tag{4}$$

where $m_s = \pi\gamma/4$ is the mass of the cylinder, γ is the cylinder to fluid density ratio, v_c is the vertical velocity (falling velocity) of the cylinder, $m_f = \pi/4$ is the mass of the displaced fluid, $g^* = \mathtt{gD}/\mathtt{U}^2 = 1$, and F_f is the vertical fluid force.

During the lubrication phase, the fluid force F_f can be obtained by integrating the pressure gradient distribution in equation 3b twice[9]

$$F_f = -\frac{3\pi\nu^*}{2} v_c h_m^{-3/2} \ , \tag{5}$$

and equation 4 becomes

$$\frac{dv_c}{dt} = -\frac{\gamma - 1}{\gamma} - \frac{6\nu^*}{\gamma} v_c h_m^{-3/2} \ . \tag{6}$$

III. ANALYTIC SOLUTIONS

Since $\frac{dh_m}{dt} = v_c$, equation 6 can be rewritten as

$$\frac{d^2 h_m}{dt^2} = -\frac{\gamma - 1}{\gamma} - \frac{6\nu^*}{\gamma Re} h_m^{-3/2} \frac{dh_m}{dt} \ . \tag{7}$$

We can integrate equation 7 once with respect to time from t_0 to t to obtain

$$v_c - v_{c,0} = -\frac{\gamma - 1}{\gamma}(t - t_0) + \frac{12\nu^*}{\gamma} \left(\frac{1}{\sqrt{h_m}} - \frac{1}{\sqrt{h_{m,0}}} \right) \ , \tag{8}$$

where $h_{m,0}$ and $v_{c,0}$ are the values of h_m and v_c at time t_0. Replacing v_c by $\frac{dh_m}{dt}$, Equation 8 can be integrated numerically using an ODE solver for $h_m(t)$, which then gives $v_c(t)$ by equation 8 and the acceleration $a(t) = \frac{d^2 h_m}{dt^2}$ by equation 7.

Setting $v_c = 0$ in equation 8, and denoting the values of h_m and t corresponding to $v_c = 0$ as $h_{m,\infty}$ and t_∞, respectively, we have

$$\frac{h_{m,\infty}}{h_{m,0}} = \frac{1}{\left\{ 1 + \frac{\gamma\sqrt{h_{m,0}}}{12\nu^*} \left[-v_{c,0} + \frac{\gamma - 1}{\gamma}(t_\infty - t_0) \right] \right\}^2} \ . \tag{9}$$

In the absence of gravity, i.e. without the first term at the right-hand side of equation 7, equation 9 becomes

$$\frac{h_{m,\infty}}{h_{m,0}} = \frac{1}{\left(1 + \frac{S\sqrt{h_{m,0}}}{12} \right)^2} \ , \tag{10}$$

where $S == \gamma(-v_{c,0})/\nu^*$; and equation 7 can be solved analytically with the initial conditions $v_c = v_{c,0}$ and $h_m = h_{m,0}$ at $t = t_0$ to give

$$\begin{aligned} t - t_0 = &- K(h_m - h_{m,0}) - 2KH_0(\sqrt{h_m} - \sqrt{h_{m,0}}) \\ &- 2KH_0^2 \ln\left(\frac{\sqrt{h_m} - H_0}{\sqrt{h_{m,0}} - H_0} \right), \end{aligned} \tag{11}$$

where $K = \frac{\gamma\sqrt{h_{m,0}}}{12\nu^* + \gamma(-v_{c,0})\sqrt{h_{m,0}}}$ and $H_0 = \frac{12\sqrt{h_{m,0}}}{12 + S\sqrt{h_{m,0}}}$. The parameter $S = \gamma(-v_{c,0})/\nu^*$ for the cylinder in 2D is like the Stokes number for a sphere in 3D. It characterizes the magnitude of the inertia of the cylinder relative to the viscous force on the cylinder. Equation 11 implies that as $t \to \infty$, $h_m \to h_{m,\infty} = H_0^2$, the same result as equation 10. So $h_{m,\infty}$ can not be reached in finite time.

IV. NUMERICAL SOLUTIONS

The dynamics of the cylinder approaching the wall are described by the minimum gap height $h_m(t)$, the approaching velocity $v_c(t)$ and the approaching acceleration $a_c(t)$ as functions of the time t. The controlling parameters are γ and γRe in the presence of gravity and only γRe in the absence of gravity. The dynamics are also affected by the initial velocity $v_{c,0}$ and the initial minimum gap $h_{m,0}$.

We use the ODE solver `ode45` in `Matlab` to numerically integrate equation 8 to obtain h_m, v_c and a_c. Without gravity, we can obtain their analytical results using equation 11. The ODE solver is validated by comparing numerical and analytical results for a non-gravity case, as shown in figure ??. As equations 11 and 10 imply, $v_c \to 0$ as $t \to \infty$. The numerical solver works until roundoff error sets in, which shows up in v_c at the large t.

A. The effect of gravity

We first investigate the role of gravity on the dynamics of a settling cylinder. To contract the two cases, we use relatively large values for Re and γ to reduce the effect of the viscous force and to increase the gravitational force.

Figure ?? compares the time history of the gap height, velocity and acceleration of the cylinder in the presence and in the absence of gravity with the same controlling parameters ($Re = 200$ and $\gamma = 10$) and initial conditions. At a given time t, the gap height is smaller in the presence of gravity. Without gravity, equations 11 and 10 imply that $v_c \to 0$ and h_m approaches a finite positive value as $t \to \infty$, and the contact ($h_m = 0$) never happens. However, with gravity, h_m keeps decreasing with time and approaches 0 as $t \to \infty$, as indicated by equation 9, and the contact cannot occur in finite time. The the cylinder cannot come to rest at a separation from the wall in finite time either, since otherwise, there would not be a fluid force to balance the weight. The numerical results in figure ?? are consistent with the theory. Figure ?? also indicates that, With or without gravity, the dynamics undergoes sharper changes in a short time for a larger initial velocity.

Without the gravity, the only controlling parameter for the dynamics of the cylinder is γRe. Figure ?? shows the dynamics in the absence of gravity as we vary γRe but keep the initial conditions the same. The higher γRe, the smaller the gap height h_m is at a given time t. Equation 10 indicates that $h_{m,\infty}$ (the value of h_m when $v_c = 0$) is a decreasing function of $S = \gamma Re(-v_{c,0})$. Figure ?? also demonstrates that there is an abrupt change of h_m with t shortly after $t = 0$, and the change is delayed as γRe increases.

The effects of initial conditions on the dynamics in the absence of gravity are reflected in figures ?? and ??. With larger initial speed or smaller initial gap height, the cylinder approaches the wall closer. As mentioned above, there is an abrupt change of h_m with t shortly after $t = 0$. Varying the initial speed has little effect on the timing of this changes, but the increase of the initial gap delays it.

B. Comparison of a cylinder and a sphere

In this study, we compare the dynamics of a cylinder and a sphere to look at the effect of the space dimension.

We analyze the dynamics of a sphere approaching a wall in a similar way as a cylinder. We now use the diameter D of the sphere as the length scale. Under the lubrication limit, the dynamics of the sphere is governed by the following ODE

$$m_s \frac{dv_c}{dt} = -(m_s - m_f)g^* - \frac{3\pi v_c}{2Reh_m} \ , \tag{12}$$

where the non-dimensional mass of the sphere is $m_s = \gamma\pi/6$, $m_f = \pi/6$ is the mass of the displaced fluid, and the fluid force is inversely proportional to h_m[1]. It can be written in terms of h_m as

$$\frac{d^2h_m}{dt^2} = \frac{\gamma - 1}{\gamma} - \frac{9}{\gamma Reh_m}\frac{dh_m}{dt} \ , \tag{13}$$

which can be integrated once with respect to time t to obtain

$$v_c - v_{c,0} = -\frac{\gamma - 1}{\gamma}(t - t_0) - \frac{9}{\gamma Re}\ln\left(\frac{h_m}{h_{m,0}}\right) \ . \tag{14}$$

To get the gap height $h_{m,\infty}$ when the sphere comes to rest, we set $v_c = 0$ in equation 14 and get

$$\frac{h_{m,\infty}}{h_{m,0}} = e^{-\frac{\gamma Re}{9}\left[-v_{c0} + \frac{\gamma-1}{\gamma}(t_\infty - t_0)\right]} \ , \tag{15}$$

In the absence of the gravity, we have

$$\frac{h_{m,\infty}}{h_{m,0}} = e^{-St} \ , \tag{16}$$

where $St = \gamma Re(-v_{c,0})/9 = \mathtt{m}\mathtt{v}_{\mathtt{c},0}/(3\pi\mu_{\mathtt{f}}\mathtt{D})$ ($\mathtt{m}$ and $\mathtt{v}_{\mathtt{c},0}$ are the dimensional mass and initial velocity of the sphere) is the Stokes number of the sphere, which is defined as the ratio of the particle inertia to the Stokes drag on the particle.

Equations 10 and 16 indicate that in the absence of gravity, $h_{m,\infty}/h_{m,0}$ decays algebraically with $S = \gamma Re(-v_{c,0})$ and is dependent on the initial gap height $h_{m,0}$ for a cylinder, while exponentially with $St = \gamma Re(-v_{c,0})/9$ and independent on $h_{m,0}$ for a sphere.

Figure?? shows the comparison of a cylinder and a sphere in terms of the dependence of $h_{m,\infty}/h_{m,0}$ on $S = \gamma Re(-v_{c,0})$. For a cylinder, it takes a much larger $S = \gamma Re(-v_{c,0})$ before micro-scale effects (surface roughness, van der Waals forces, and non-continuum effects etc.) set in. As such, the lubrication approximations for a cylinder is valid for a larger range of density ratios and Reynolds numbers.

In figure **??** we compare the dynamics of a cylinder and a sphere approaching a wall in the presence of gravity with the same γ, Re and initial conditions. For a cylinder, it takes much longer time before micro-scale effects set in, and the lubrication approximations for a cylinder is valid for a longer time.

C. Simulations of full dynamics with lubrication theory incorporated to handle the collision

We have developed a flow simulation method to couple the dynamics of a fluid and moving particles in both the inertia and lubrication phases of the particles. The underlying direct numerical simulation method is the immersed interface method[17,18,19], which enforces boundary conditions by singular forces at the boundaries of particles. In the inertia phase when a particle is away from the others, a boundary-condition-capturing strategy[19] is used to numerically determine the singular forces. When two particles are in close proximity, we use the lubrication approximations to analytically determine the singular forces in the lubrication region. Detailed description and validation of the method will be reported elsewhere.

Figure **??** shows the vorticity contours around a cylinder settling under gravity toward a fixed wall. The flow is in the inertia regime in the first four snapshots, and in the lubrication regime in the last one.

V. CONCLUSIONS

We investigated the detailed dynamic behaviors of a cylinder approaching a wall in a viscous fluid. We showed that without gravity, the cylinder comes to rest asymptotically at a finite separation from the wall, and contact does not occur, while if there is a gravity pulling the cylinder toward the wall, the cylinder approaches the wall as time goes to infinity, and the particle does not rest or contact with the wall in finite time. We found that when the initial inertia of the cylinder is high relative to the viscous fluid force, i.e. when the product of the density ratio, the Reynolds number and the initial velocity is large, the gap height quickly undergoes a sharp decrease with time in the presence or absence of gravity, and the time history of the approaching velocity in the presence of gravity appears like a step function.

We compared the dynamics of a cylinder in 2D with a sphere in 3D. The above conclusions for a cylinder in 2D also apply to a sphere in 3D. The main difference lies in the rates of decay. Without gravity, the asymptotic gap height decays with the Stokes number algebraically for a cylinder while exponentially for a sphere. Additionally, the gap height for a cylinder decreases with time slower than a sphere under the same conditions with or without gravity. So the lubrication theory, before its breakdown by micro-scale effects such as surface roughness, van der Waals forces, and non-continuum effects etc., is valid for a longer time and a wider range of parameters in the case of a cylinder.

Finally, we demonstrated that the lubrication approximations can be incorporated into a flow simulation method to couple the dynamics of the fluid and particles in both inertia-dominant regime and viscosity-dominant regime. This offers a computational tool for examining a large range of phenomena such as rain droplet formation.

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