A MATRIX BASIS FORMULATION FOR THE DYADIC GREEN'S FUNCTIONS OF MAXWELL'S EQUATIONS IN LAYERED MEDIA

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Abstract. A matrix basis formulation is introduced to represent the 3×3 frequency domain dyadic Green's functions of the Maxwell's equations in layered media. The formulation can be used to decompose the Maxwell's Green's functions into independent TE and TM components, each satisfying a scalar Helmholtz equation. Moreover, the interface transmission conditions for the electromagnetic waves can be reduced to decoupled conditions for the matrix basis coefficients at interfaces. Numerical results for the electric and magnetic dyadic Green's functions for a 10-layer medium validate the accuracy of the proposed formulation.

Key words. Dyadic Green's functions, Maxwell's equations, Layered media, Matrix basis.

AMS subject classifications. 15A15, 15A09, 15A23

1. Introduction. Layered media dyadic Green's functions (LMDG) of the Maxwell's equations are commonly used in the integral equation methods for studying wave fields in layered media [12, 9, 11]. These Green's functions are 3×3 tensors which satisfy the Maxwell equations or their variants and certain physical transmission conditions across interfaces between layers. A naive derivation of the Green's functions will have 9 unknown entries of the 3×3 tensor in each layer whereas the transmission conditions will tangle all the entries together. However, some of the entries are in fact linearly dependent or even identical. To simplify the derivation and reduce computational cost, a number of formulations of the Maxwell's LMDG have been proposed, such as formulations using the Sommerfeld potential [1], the transverse potential [2, 4] as well as the Michalski–Zheng formulations [5], the E_z-H_z formulation [3, 7], etc. The Sommerfeld potential and the transverse potential formulations reduce the number of unknowns to 5 while the E_z-H_z approach uses merely 2 scalar variables, based on a TE/TM mode decomposition.

The purpose of this paper is to present a theoretically sound general matrix representation of the 3×3 LMDG of the Maxwell's equations using a linear matrix basis, providing an alternative formulation, in comparison with some previously known results [3, 7]. Moreover, this same matrix basis will be applied to study the dyadic Green's function of the elastic wave equations, which will be the subject of a follow-up paper. It will be shown that there are several remarkable benefits resulting from the matrix basis formulation (MBF). First, the coefficients of the matrix basis are all radial symmetric in the horizontal directions, so that the evaluation of the reflection/transmission coefficients in the layers are simplified. Second, the Maxwell's Green's functions can be naturally decomposed into independent TE and TM components within this formulation, leading to the 2-term E_z-H_z result [3, 7].Third, the radial symmetry allows us to apply fast solvers easily, e.g. the fast multipole method in layered media [13].

The rest of this paper is organized as follows. In Section 2, we establish the theo-

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ries of the matrix basis and propose the MBF. In Section 3, the details of the Maxwell's LMDG are derived, including a 5-term matrix-based general formulation and the concise 2-term formulation. In Section 4, the electric and magnetic field LMDGs for a 10-layer problem are numerically calculated using the formulation proposed in this paper. A conclusion and discussion on future work are given in Section 5.

2. A matrix basis formulation. In this section, we set up the matrix basis used for the LMDG of the Maxwell's equations, and develop basic theories regarding the basis coefficients.

Consider a layered medium with $L+1$ layers indexed by $0, \cdots, L$ from top to bottom. The interfaces are given by $z = d_l$, $0 \leq l \leq L-1$. Suppose $\{k_i\}_{i=1}^{I}$ representing all the wave numbers in the layers. The interaction between a given source $\mathbf{r}' = (x', y', z')$ and a target $\mathbf{r} = (x, y, z)$ will be studied, assuming none of them resides on an interface.

Using the 2-D Fourier transform for a function $f(x, y)$ from $(x - x', y - y')$ to (k_x, k_y), we have the following representation

$$f(x,y) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} e^{\mathrm{i}k_x(x-x') + \mathrm{i}k_y(y-y')} \widehat{f}(k_x, k_y) dk_x dk_y. \tag{2.1}$$

Let (k_ρ, α) be the polar coordinates of (k_x, k_y), i.e.

$$k_x = k_\rho \cos\alpha, \quad k_y = k_\rho \sin\alpha, \quad k_\rho \in \overline{\mathbb{R}^+} = [0, \infty), \quad \alpha \in [0, 2\pi). \tag{2.2}$$

2.1. Function fields for the LMDGs. We will construct some fields of functions that will be helpful representing LMDGs. For the sake of convenience, define

$$R_0 = [0, \infty) \setminus \{k_i : 1 \leq i \leq I\}. \tag{2.3}$$

Using the field of complex numbers $\mathbb{C}$, define

$$\mathbb{F}_0 = \left\{ f_0(k_\rho) = \frac{p_0\,(s_i, e_{ij}, e_{iz}, e_{iz'})}{q_0\,(s_i, e_{ij}, e_{iz}, e_{iz'})} : p_0, q_0 \in \mathbb{P}[\mathbb{C}], q_0\,(s_i, e_{ij}, e_{iz}, e_{iz'}) \not\equiv 0 \right\}, \tag{2.4}$$

where

$$s_i = \mathrm{i}\sqrt{k_i^2 - k_\rho^2}, \;\; e_{ij} = e^{\mathrm{i}\sqrt{k_i^2 - k_\rho^2} d_j}, \;\; e_{iz} = e^{\mathrm{i}\sqrt{k_i^2 - k_\rho^2} z}, \;\; e_{iz'} = e^{\mathrm{i}\sqrt{k_i^2 - k_\rho^2} z'} \tag{2.5}$$

are continuous complex functions of k_ρ defined on R_0, in (2.4) indices ranging in $1 \leq i \leq I$, $0 \leq j \leq L-1$ are considered, and $\mathbb{P}[\mathbb{C}]$ is the collection of polynomials with coefficients in $\mathbb{C}$. Although the natural domain of any function f_0 in $\mathbb{F}_0$ may exclude zeros of the denominator in the definition of (2.4), we claim that the domain of f_0 can always be extended to R_0 while ranging in the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Consider an analytic extension of the denominator $q_0\,(s_i, e_{ij}, e_{iz}, e_{iz'})$ as a function of k_ρ with a series of branch cuts (see subsection 2.1)

$$B = \bigcup_{i=1}^{I} \{\pm(k_i - \ell \mathrm{i}) : \ell \in [0, \infty)\} \tag{2.6}$$

which leaves the remaining part of the complex plane $\mathbb{C} \setminus B$ connected. By the identity theorem of the complex analysis, we conclude that the denominator is either identical to 0 (which is excluded from the definition), or having *isolated* zeros in $\mathbb{C} \setminus B$

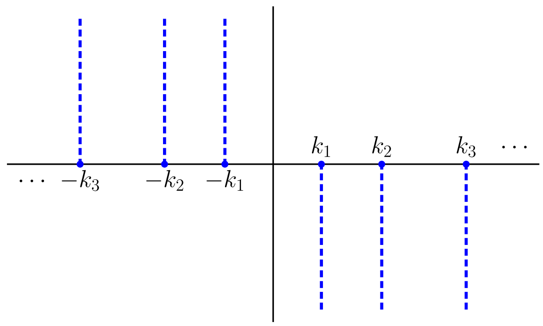


Fig. 2.1: Branch cuts defined by (2.6).

(hence also isolated in R_0). The same properties are also applied to the numerator $p_0\left(s_i, e_{ij}, e_{iz}, e_{iz'}\right)$. Therefore, we can extend the domain of f_0 to R_0 by taking the limit from the neighborhoods of each zero.

When treating any element of $\mathbb{F}_0$ as a mapping from R_0 to $\overline{\mathbb{C}}$, $\mathbb{F}_0$ forms a field. The proof is trivial.

Proposition 2.1 (Field structure of $\mathbb{F}_0$). *$\mathbb{F}_0$ is a field of functions mapping R_0 to $\overline{\mathbb{C}}$, with function addition and multiplication, where the constant functions 0 and 1 are the additive identity and the multiplicative identity, respectively.*

It is worth mentioning that $k_\rho^2 \in \mathbb{F}_0$ because

$$k_\rho^2 = k_i^2 + s_i^2. \tag{2.7}$$

Also, we can treat elements of $\mathbb{F}_0$ as functions of k_x and k_y, with the relationship $k_\rho = \sqrt{k_x^2 + k_y^2}$.

Next, define

$$\mathbb{F} = \left\{ f(\mathrm{i}k_x, \mathrm{i}k_y) = \frac{p(\mathrm{i}k_x, \mathrm{i}k_y)}{q(\mathrm{i}k_x, \mathrm{i}k_y)} : p, q \in \mathbb{P}[\mathbb{F}_0], q(\mathrm{i}k_x, \mathrm{i}k_y) \not\equiv 0 \right\}, \tag{2.8}$$

where $\mathbb{P}[\mathbb{F}_0]$ is the collection of polynomials with coefficients in $\mathbb{F}_0$. It is clear that $\mathbb{F}$ is a field extension of $\mathbb{F}_0$ with elements $\mathrm{i}k_x$ and $\mathrm{i}k_y$. We will deduce similar results of $\mathbb{F}$, but with the polar coordinates. Given any $f \in \mathbb{F}$, let $D_f \subset \mathbb{C} \setminus B$ be the set of any k_ρ that makes ∞ exist in coefficients of polynomials p and q. D_f is known as a discrete set according to the previous discussion on $\mathbb{F}_0$. While also excluding the branch cuts B defined in (2.6) from the complex plane, we have an analytic extension of

$$q(\mathrm{i}k_x, \mathrm{i}k_y) = q(\mathrm{i}k_\rho \cos\alpha, \mathrm{i}k_\rho \sin\alpha) := \tilde{q}(k_\rho, \alpha) \tag{2.9}$$

for (k_ρ, α) in the open connected domain $\mathbb{C} \setminus (B \cup D_f) \times \mathbb{C}$. A simple generalization of the identity theorem is sufficient to tell that $\tilde{q}(k_\rho, \alpha)$ is either identical to 0 or having isolated zeros in this domain. Therefore, similarly, any function $f \in \mathbb{F}$ can be extended to a mapping from $(k_\rho, \alpha) \in R_0 \times [0, 2\pi)$ to $\overline{\mathbb{C}}$, and $\mathbb{F}$ is a field in this sense.

Remark 2.2. The coordinates z and z' are in fact redundant in the definition of $\mathbb{F}_0$ and $\mathbb{F}$ for the matrix basis theory and they are included only for the convenience of statements in later sections.

2.2. The matrix basis. One of our expectations for the matrix basis is to represent the tensor Green's functions with all coefficients belonging to $\mathbb{F}_0$, i.e., information of the polar angle α would only appear in the matrix basis. For this purpose, based on an observation of derived formulas for the Maxwell's Green's functions in a 3-layer problem [10], we propose a matrix basis $\mathbf{J}_1, \cdots, \mathbf{J}_9$ in the frequency domain as follows.

PROPOSITION 2.3 (The matrix basis). *These matrices form a basis of* $\mathbb{F}^{3\times 3}$*:*

$$\begin{aligned}
&\mathbf{J}_1 = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \ \mathbf{J}_2 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}, \ \mathbf{J}_3 = \begin{bmatrix} 0 & 0 & \mathrm{i}k_x \\ 0 & 0 & \mathrm{i}k_y \\ 0 & 0 & 0 \end{bmatrix}, \\
&\mathbf{J}_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{i}k_x & \mathrm{i}k_y & 0 \end{bmatrix}, \ \mathbf{J}_5 = \begin{bmatrix} -k_x^2 & -k_xk_y & 0 \\ -k_xk_y & -k_y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{J}_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\mathrm{i}k_y & \mathrm{i}k_x & 0 \end{bmatrix}, \\
&\mathbf{J}_7 = \begin{bmatrix} 0 & 0 & \mathrm{i}k_y \\ 0 & 0 & -\mathrm{i}k_x \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{J}_8 = \begin{bmatrix} k_xk_y & k_y^2 & 0 \\ -k_x^2 & -k_xk_y & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{J}_9 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned} \tag{2.10}$$

The proof is trivial.

The above matrix basis will be divided into two groups according to the product properties between them. For any subfield $\mathbb{K} \subset \mathbb{F}$, we define the following vector spaces with coefficients in $\mathbb{K}$

$$\begin{aligned}
\mathfrak{R}(\mathbb{K}) &= \mathrm{span}_{\mathbb{K}}(\mathbf{J}_1, \cdots, \mathbf{J}_5), \\
\mathfrak{I}(\mathbb{K}) &= \mathrm{span}_{\mathbb{K}}(\mathbf{J}_6, \cdots, \mathbf{J}_9), \\
\mathfrak{M}(\mathbb{K}) &= \mathrm{span}_{\mathbb{K}}(\mathbf{J}_1, \cdots, \mathbf{J}_5, \mathbf{J}_6, \cdots, \mathbf{J}_9).
\end{aligned} \tag{2.11}$$

PROPOSITION 2.4. *Let* $\mathbb{K}$ *be any subfield of* $\mathbb{F}$ *containing* k_ρ^2*. Then, we have*

- $\mathfrak{M}(\mathbb{K}) = \mathfrak{R}(\mathbb{K}) \oplus \mathfrak{I}(\mathbb{K})$ *is the direct sum.*
- $\mathfrak{R}(\mathbb{K})$*,* $\mathfrak{M}(\mathbb{K})$ *are rings with matrix addition and matrix multiplication.*

Proof. The direct sum is obvious. For the ring property, notice the identity matrix

$$\mathbf{I} = \mathbf{J}_1 + \mathbf{J}_2 \in \mathfrak{R}(\mathbb{K}) \subset \mathfrak{M}(\mathbb{K}), \tag{2.12}$$

and the product table of the matrices $\mathbf{J}_1, \cdots, \mathbf{J}_9$

$$\begin{aligned}
&\begin{bmatrix} \mathbf{J}_1^T & \cdots & \mathbf{J}_9^T \end{bmatrix}^T \cdot \begin{bmatrix} \mathbf{J}_1 & \cdots & \mathbf{J}_9 \end{bmatrix} = \\
&\left[\begin{array}{ccccc:cccc}
\mathbf{J}_1 & \mathbf{0} & \mathbf{J}_3 & \mathbf{0} & \mathbf{J}_5 & \mathbf{0} & \mathbf{J}_7 & \mathbf{J}_8 & \mathbf{J}_9 \\
\mathbf{0} & \mathbf{J}_2 & \mathbf{0} & \mathbf{J}_4 & \mathbf{0} & \mathbf{J}_6 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{J}_3 & \mathbf{0} & \mathbf{J}_5 & \mathbf{0} & \mathbf{J}_8 - k_\rho^2\mathbf{J}_9 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{J}_4 & \mathbf{0} & -k_\rho^2\mathbf{J}_2 & \mathbf{0} & -k_\rho^2\mathbf{J}_4 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}_6 \\
\mathbf{J}_5 & \mathbf{0} & -k_\rho^2\mathbf{J}_3 & \mathbf{0} & -k_\rho^2\mathbf{J}_5 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{J}_8 - k_\rho^2\mathbf{J}_9 \\
\hdashline
\mathbf{J}_6 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & k_\rho^2\mathbf{J}_2 & -k_\rho^2\mathbf{J}_4 & -\mathbf{J}_4 \\
\mathbf{0} & \mathbf{J}_7 & \mathbf{0} & -\mathbf{J}_8 & \mathbf{0} & k_\rho^2\mathbf{J}_1 + \mathbf{J}_5 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{J}_8 & \mathbf{0} & k_\rho^2\mathbf{J}_7 & \mathbf{0} & -k_\rho^2\mathbf{J}_8 & \mathbf{0} & \mathbf{0} & \mathbf{0} & -k_\rho^2\mathbf{J}_1 - \mathbf{J}_5 \\
\mathbf{J}_9 & \mathbf{0} & \mathbf{J}_7 & \mathbf{0} & -\mathbf{J}_8 & \mathbf{0} & -\mathbf{J}_3 & \mathbf{J}_5 & -\mathbf{J}_1
\end{array}\right],
\end{aligned} \tag{2.13}$$

which ensures the matrix multiplication is closed in both $\mathfrak{M}(\mathbb{K})$ and $\mathfrak{R}(\mathbb{K})$. ∎

The product table (2.13) immediately implies the product rules below.

Proposition 2.5 (The product rules). *Let $\mathbb{K}$ be any subfield of $\mathbb{F}$ containing k_ρ^2.*

- *If* $\mathbf{A} \in \mathfrak{R}(\mathbb{K})$, $\mathbf{B} \in \mathfrak{R}(\mathbb{K})$, *then* $\mathbf{A} \cdot \mathbf{B} \in \mathfrak{R}(\mathbb{K})$.
- *If* $\mathbf{A} \in \mathfrak{R}(\mathbb{K})$, $\mathbf{B} \in \mathfrak{I}(\mathbb{K})$, *then* $\mathbf{A} \cdot \mathbf{B} \in \mathfrak{I}(\mathbb{K})$.
- *If* $\mathbf{A} \in \mathfrak{I}(\mathbb{K})$, $\mathbf{B} \in \mathfrak{R}(\mathbb{K})$, *then* $\mathbf{A} \cdot \mathbf{B} \in \mathfrak{I}(\mathbb{K})$.
- *If* $\mathbf{A} \in \mathfrak{I}(\mathbb{K})$, $\mathbf{B} \in \mathfrak{I}(\mathbb{K})$, *then* $\mathbf{A} \cdot \mathbf{B} \in \mathfrak{R}(\mathbb{K})$.

The behavior in the product rules resembles the products between the real numbers and the pure imaginary numbers, which explains the adopted letters $\mathfrak{R}$ and $\mathfrak{I}$.

Definition 2.6 (An $\mathfrak{R}^0$-matrix basis formulation). *Define the linear space*

$$\mathfrak{R}^0 = \operatorname{span}_{\mathbb{F}_0}\{\mathbf{J}_1, \cdots, \mathbf{J}_5\} = \left\{ \sum_{j=1}^{5} a_j \mathbf{J}_j : a_j \in \mathbb{F}_0 \right\}. \tag{2.14}$$

The linear expansion of functions in $\mathfrak{R}^0$ with matrix basis $\mathbf{J}_1, \cdots, \mathbf{J}_5$ will be defined as the $\mathfrak{R}^0$-matrix basis formulation.

In the coming sections, we will show that this MBF can be used to efficiently express the Green's functions for the Maxwell's equations in layered media. Since the tensors Green's functions are coupled between the layers, we introduce the framework and theories of the block matrices.

For any subfield $\mathbb{K} \subset \mathbb{F}$ and any $p, q \in \mathbb{N}$, define the linear spaces of block matrices

$$\begin{aligned}
\mathfrak{M}_{p\times q}(\mathbb{K}) &= \left\{ \sum_{j=1}^{9} \mathbf{K}_j \otimes \mathbf{J}_j : \mathbf{K}_j \in \mathbb{K}^{p\times q}, 1 \le j \le 9 \right\}, \\
\mathfrak{R}_{p\times q}(\mathbb{K}) &= \left\{ \sum_{j=1}^{5} \mathbf{K}_j \otimes \mathbf{J}_j : \mathbf{K}_j \in \mathbb{K}^{p\times q}, 1 \le j \le 5 \right\}, \\
\mathfrak{I}_{p\times q}(\mathbb{K}) &= \left\{ \sum_{j=6}^{9} \mathbf{K}_j \otimes \mathbf{J}_j : \mathbf{K}_j \in \mathbb{K}^{p\times q}, 6 \le j \le 9 \right\},
\end{aligned} \tag{2.15}$$

where $\otimes$ is the Kronecker product. Any $\sum_{j=1}^{9} \mathbf{K}_j \otimes \mathbf{J}_j \in \mathfrak{M}_{p\times q}(\mathbb{K})$ is a $3p \times 3q$ matrix consists of 3×3 blocks in $\mathfrak{M}(\mathbb{K})$. By applying the direct sum decomposition from Proposition 2.4 to each 3×3 block, we get the direct sum decomposition of the block matrices

$$\mathfrak{M}_{p\times q}(\mathbb{K}) = \mathfrak{R}_{p\times q}(\mathbb{K}) \oplus \mathfrak{I}_{p\times q}(\mathbb{K}). \tag{2.16}$$

Moreover, the product rules for block matrices are easily generalized as follows.

Proposition 2.7 (The product rules for block matrices). *Let $p, q, r \in \mathbb{N}$. Let $\mathbb{K}$ be any subfield of $\mathbb{F}$ containing k_ρ^2.*

- *If* $\bar{\mathbf{A}} \in \mathfrak{R}_{p\times r}(\mathbb{K})$, $\bar{\mathbf{B}} \in \mathfrak{R}_{r\times q}(\mathbb{K})$, *then* $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}} \in \mathfrak{R}_{p\times q}(\mathbb{K})$.
- *If* $\bar{\mathbf{A}} \in \mathfrak{R}_{p\times r}(\mathbb{K})$, $\bar{\mathbf{B}} \in \mathfrak{I}_{r\times q}(\mathbb{K})$, *then* $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}} \in \mathfrak{I}_{p\times q}(\mathbb{K})$.
- *If* $\bar{\mathbf{A}} \in \mathfrak{I}_{p\times r}(\mathbb{K})$, $\bar{\mathbf{B}} \in \mathfrak{R}_{r\times q}(\mathbb{K})$, *then* $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}} \in \mathfrak{I}_{p\times q}(\mathbb{K})$.
- *If* $\bar{\mathbf{A}} \in \mathfrak{I}_{p\times r}(\mathbb{K})$, $\bar{\mathbf{B}} \in \mathfrak{I}_{r\times q}(\mathbb{K})$, *then* $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}} \in \mathfrak{R}_{p\times q}(\mathbb{K})$.

Proof. We will only take the second proposition as an example. Suppose

$$\bar{\mathbf{A}} = \sum_{j=1}^{5} \mathbf{A}_j \otimes \mathbf{J}_j, \quad \bar{\mathbf{B}} = \sum_{l=6}^{9} \mathbf{B}_l \otimes \mathbf{J}_l, \quad \mathbf{A}_j \in \mathbb{K}^{p\times r}, \quad \mathbf{B}_l \in \mathbb{K}^{r\times q}, \tag{2.17}$$

The matrix product is given by

$$\bar{\mathbf{A}} \cdot \bar{\mathbf{B}} = \sum_{j=1}^{5} \sum_{l=6}^{9} (\mathbf{A}_j \mathbf{B}_l) \otimes (\mathbf{J}_j \mathbf{J}_l), \tag{2.18}$$

where each $\mathbf{A}_j \mathbf{B}_l \in \mathbb{K}^{p\times q}$ and each $\mathbf{J}_j \mathbf{J}_l \in \mathfrak{I}(\mathbb{K})$ by the product table (2.13). Therefore, $\bar{\mathbf{A}} \cdot \bar{\mathbf{B}}$ is a $3p \times 3q$ block matrix with every block in $\mathfrak{I}(\mathbb{K})$. ∎

PROPOSITION 2.8. *For any $p, q \in \mathbb{N}$, $\mathfrak{M}_{p\times q}(\mathbb{F}) = \mathbb{F}^{3p\times 3q}$.*

The proof is trivial with the fact that $\mathbf{J}_1, \cdots, \mathbf{J}_9$ form a basis of $\mathbb{F}^{3\times 3}$.

2.3. The solution filtering theorem. Just like the fact that any solvable real linear system with a complex solution must also have a real one, the following theorem claims that when the coefficient matrices have the $\mathfrak{R}^0$-matrix basis representation for each of their 3×3 blocks, there exists a solution with an $\mathfrak{R}^0$-matrix basis representation as well, when solvable.

We begin with a well-known lemma for linear systems of fields.

LEMMA 2.9 (Solution filtering of fields). *Suppose $p, q, r \in \mathbb{N}$, $\mathbb{K}$ is a field, $\mathbf{A} \in \mathbb{K}^{p\times r}$, $\mathbf{B} \in \mathbb{K}^{p\times q}$ are coefficients of a solvable linear system, i.e. $\mathbf{A} \cdot \mathbf{X} = \mathbf{B}$ for some $r \times q$ matrix $\mathbf{X}$. Then, there exists a "filtered" solution $\mathbf{X}_0 \in \mathbb{K}^{r\times q}$ such that $\mathbf{A} \cdot \mathbf{X}_0 = \mathbf{B}$.*

Proof. Let $a \leq \min(p, r)$ be the rank of $\mathbf{A}$. By applying elementary row and column operations we can transform $\mathbf{A}$ in its reduced row echelon form, then into a 0-1 matrix with $\mathbf{I}_a$ on the top left corner while all other entries are zero. Hence there exist *full-rank* matrices $\mathbf{S} \in \mathbb{K}^{p\times p}$ and $\mathbf{T} \in \mathbb{K}^{r\times r}$ such that

$$\mathbf{A} = \mathbf{S} \cdot \begin{bmatrix} \mathbf{I}_a & \mathbf{0}_{a\times(r-a)} \\ \mathbf{0}_{(p-a)\times a} & \mathbf{0}_{(p-a)\times(r-a)} \end{bmatrix} \cdot \mathbf{T}. \tag{2.19}$$

It immediately follows that $\mathbf{S}^{-1}\mathbf{B} \in \mathbb{K}^{p\times q}$ is a matrix whose bottom $(p-a)$ rows are all zero. Next, we define

$$\mathbf{X}_0 = \mathbf{T}^{-1} \cdot \begin{bmatrix} \begin{bmatrix} \mathbf{I}_a & \mathbf{0}_{a\times(p-a)} \end{bmatrix} \cdot \left(\mathbf{S}^{-1}\mathbf{B}\right) \\ \mathbf{0}_{(r-a)\times q} \end{bmatrix} \in \mathbb{K}^{r\times q}, \tag{2.20}$$

then we have

$$\mathbf{A}\mathbf{X}_0 = \mathbf{S} \begin{bmatrix} \begin{bmatrix} \mathbf{I}_a & \mathbf{0}_{a\times(p-a)} \end{bmatrix} \cdot \left(\mathbf{S}^{-1}\mathbf{B}\right) \\ \mathbf{0}_{(p-a)\times q} \end{bmatrix} \tag{2.21}$$

and

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{X}_0 = \begin{bmatrix} \begin{bmatrix} \mathbf{I}_a & \mathbf{0}_{a\times(p-a)} \end{bmatrix} \cdot \left(\mathbf{S}^{-1}\mathbf{B}\right) \\ \mathbf{0}_{(p-a)\times q} \end{bmatrix}. \tag{2.22}$$

We can see that $\mathbf{S}^{-1}\mathbf{A}\mathbf{X}_0$ has its top a rows identical to those of $\mathbf{S}^{-1}\mathbf{B}$, and its bottom $(p-a)$ rows are zero entries (namely, exactly as $\mathbf{S}^{-1}\mathbf{B}$ as noted above). Therefore, $\mathbf{S}^{-1}\mathbf{A}\mathbf{X}_0$ and $\mathbf{S}^{-1}\mathbf{B}$ are identical, so $\mathbf{A}\mathbf{X}_0 = \mathbf{B}$. ∎

The main result of this section is an improved version which preserves the matrix basis structure.

THEOREM 2.10 (Solution filtering). *Suppose $p, q, r \in \mathbb{N}$, $\bar{\mathbf{A}} \in \mathfrak{R}_{p\times r}(\mathbb{F}_0)$ and $\bar{\mathbf{B}} \in \mathfrak{R}_{p\times q}(\mathbb{F}_0)$ are coefficients of a solvable linear system, i.e. $\bar{\mathbf{A}} \cdot \bar{\mathbf{X}} = \bar{\mathbf{B}}$ for some $3r \times 3q$ matrix $\bar{\mathbf{X}}$. Then, there exists a "filtered" version of block matrix solution $\bar{\mathbf{X}}_0 \in \mathfrak{R}_{r\times q}(\mathbb{F}_0)$, i.e., each 3×3 block of $\bar{\mathbf{X}}_0$ has an $\mathfrak{R}^0$-matrix basis representation, so that $\bar{\mathbf{A}} \cdot \bar{\mathbf{X}}_0 = \bar{\mathbf{B}}$.*

Proof. By Proposition 2.8, $\bar{\mathbf{A}} \in \mathbb{F}^{3p\times 3r}$ and $\bar{\mathbf{B}} \in \mathbb{F}^{3p\times 3q}$. By Lemma 2.9, we can suppose $\bar{\mathbf{X}} \in \mathbb{F}^{3r\times 3q} = \mathfrak{M}_{r\times q}(\mathbb{F})$ without loss of generality.

To find the filtered solution in $\mathfrak{R}_{r\times q}(\mathbb{F}_0)$, we begin with an intermediate one in $\mathfrak{R}_{r\times q}(\mathbb{F})$. Write the direct sum decomposition $\bar{\mathbf{X}} = \bar{\mathbf{X}}_1 \oplus \bar{\mathbf{X}}_2$, where $\bar{\mathbf{X}}_1 \in \mathfrak{R}_{r\times q}(\mathbb{F})$ and $\bar{\mathbf{X}}_2 \in \mathfrak{I}_{r\times q}(\mathbb{F})$. By Proposition 2.7 we immediately get $\bar{\mathbf{A}} \cdot \bar{\mathbf{X}}_1 \in \mathfrak{R}_{p\times q}(\mathbb{F})$ and $\bar{\mathbf{A}} \cdot \bar{\mathbf{X}}_2 \in \mathfrak{I}_{p\times q}(\mathbb{F})$, so

$$\bar{\mathbf{A}} \cdot \bar{\mathbf{X}}_1 + \bar{\mathbf{A}} \cdot \bar{\mathbf{X}}_2 \tag{2.23}$$

is the direct sum decomposition of $\bar{\mathbf{B}} \in \mathfrak{R}_{p\times q}(\mathbb{F}_0) \subset \mathfrak{M}_{p\times q}(\mathbb{F})$. Therefore $\bar{\mathbf{A}} \cdot \bar{\mathbf{X}}_1 = \bar{\mathbf{B}}$.

Then, let

$$\bar{\mathbf{A}} = \sum_{j=1}^{5} \mathbf{A}_j \otimes \mathbf{J}_j, \quad \bar{\mathbf{X}}_1 = \sum_{j=1}^{5} \mathbf{X}_j^1 \otimes \mathbf{J}_j, \quad \bar{\mathbf{B}} = \sum_{j=1}^{5} \mathbf{B}_j \otimes \mathbf{J}_j,$$

where each $\mathbf{A}_j \in \mathbb{F}_0^{p\times r}$, $\mathbf{X}_j^1 \in \mathbb{F}^{r\times q}$ and $\mathbf{B}_j \in \mathbb{F}_0^{p\times q}$. When treating $\mathbf{X}_j^1$ as the solution to the linear equation $\bar{\mathbf{A}} \cdot \bar{\mathbf{X}}_1 = \bar{\mathbf{B}}$, the equation is equivalent to

$$\sum_{u=1}^{5}\sum_{v=1}^{5} (\mathbf{A}_u \mathbf{X}_v^1) \otimes (\mathbf{J}_u \mathbf{J}_v) = \sum_{j=1}^{5} \mathbf{B}_j \otimes \mathbf{J}_j, \tag{2.24}$$

where, using the product table (2.13), among the 25 products only 13 of them will be nonzero to give

$$\begin{aligned}\sum_{u=1}^{5}\sum_{v=1}^{5} (\mathbf{A}_u \mathbf{X}_v^1) \otimes (\mathbf{J}_u \mathbf{J}_v) &= (\mathbf{A}_1 \mathbf{X}_1^1) \otimes \mathbf{J}_1 + (\mathbf{A}_2 \mathbf{X}_2^1 - k_\rho^2 \mathbf{A}_4 \mathbf{X}_3^1) \otimes \mathbf{J}_2 \\ &+ \big(\mathbf{A}_3 \mathbf{X}_2^1 + (\mathbf{A}_1 - k_\rho^2 \mathbf{A}_5)\mathbf{X}_3^1\big) \otimes \mathbf{J}_3 + (\mathbf{A}_4 \mathbf{X}_1^1 + \mathbf{A}_2 \mathbf{X}_4^1 - k_\rho^2 \mathbf{A}_4 \mathbf{X}_5^1) \otimes \mathbf{J}_4 \\ &+ \big(\mathbf{A}_5 \mathbf{X}_1^1 + \mathbf{A}_3 \mathbf{X}_4^1 + (\mathbf{A}_1 - k_\rho^2 \mathbf{A}_5)\mathbf{X}_5^1\big) \otimes \mathbf{J}_5.\end{aligned}$$

By comparing the coefficients of $\mathbf{J}_1, \cdots, \mathbf{J}_5$ on each 3×3 block, the above equation (2.24) is in fact equivalent to the linear system

$$\widetilde{\mathbf{A}}\widetilde{\mathbf{X}}^1 = \widetilde{\mathbf{B}}, \tag{2.25}$$

where a *stacked* form of $\bar{\mathbf{X}}^1$ given below is used

(2.26)
$$\widetilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & -k_\rho^2 \mathbf{A}_4 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_3 & \mathbf{A}_1 - k_\rho^2 \mathbf{A}_5 & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_4 & \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & -k_\rho^2 \mathbf{A}_4 \\ \mathbf{A}_5 & \mathbf{0} & \mathbf{0} & \mathbf{A}_3 & \mathbf{A}_1 - k_\rho^2 \mathbf{A}_5 \end{bmatrix}, \quad \widetilde{\mathbf{X}}^1 = \begin{bmatrix} \mathbf{X}_1^1 \\ \vdots \\ \mathbf{X}_5^1 \end{bmatrix}, \quad \widetilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \vdots \\ \mathbf{B}_5 \end{bmatrix}.$$

Since $\widetilde{\mathbf{A}} \in \mathbb{F}_0^{5p\times 5r}$ and $\widetilde{\mathbf{B}} \in \mathbb{F}_0^{5p\times q}$, by Lemma 2.9, there exists $\widetilde{\mathbf{X}}^0 \in \mathbb{F}^{5r\times q}$ such that $\widetilde{\mathbf{A}}\widetilde{\mathbf{X}}^0 = \widetilde{\mathbf{B}}$. By writing $\widetilde{\mathbf{X}}^0$ in the stacked form

$$\widetilde{\mathbf{X}}^0 = \begin{bmatrix} \mathbf{X}_1^0 \\ \vdots \\ \mathbf{X}_5^0 \end{bmatrix} \tag{2.27}$$

where each $\mathbf{X}_j^0 \in \mathbb{F}_0^{r\times q}$, we can see that the matrix

$$\bar{\mathbf{X}}_0 = \sum_{j=1}^{5} \mathbf{X}_j^0 \otimes \mathbf{J}_j \in \mathfrak{R}_{r\times q}(\mathbb{F}_0) \tag{2.28}$$

is as desired. $\square$

3. Application to the Maxwell's equations in layered media. In this section we first give a brief introduction about the dyadic Green's functions of the time-harmonic Maxwell's equations in the free space, then we will discuss the LMDG and its simplification using the $\mathfrak{R}^0$-matrix basis formulation (2.14).

3.1. The dyadic Green's functions of the time harmonic Maxwell's equations in the free space. The source-free time harmonic Maxwell's equations in the free space is given by

$$\begin{aligned} \nabla\times\vec{E} &= -\mathrm{i}\omega\vec{B}, \qquad & \nabla\times\vec{H} &= \mathrm{i}\omega\vec{D}, \\ \nabla\cdot\vec{D} &= 0, & \nabla\cdot\vec{B} &= 0, \end{aligned} \tag{3.1}$$

where $\vec{D}(\mathbf{r})$, $\vec{E}(\mathbf{r})$ are the electric displacement flux and the electric field, $\vec{B}(\mathbf{r})$, $\vec{H}(\mathbf{r})$ are the magnetic flux density and the magnetic field, ω is the angular frequency in time. The system of Maxwell's equations is closed by the constitutive relations. In the free space, they are

$$\vec{D} = \varepsilon\vec{E}, \qquad \vec{B} = \mu\vec{H}, \tag{3.2}$$

where ε and μ are the constant permittivity and permeability in the free space. When dealing with these equations, the Lorenz gauge condition is often used, which allows us to use a vector potential $\vec{A}(\mathbf{r})$ to represent the electric field $\vec{E}$ and the magnetic field $\vec{H}$ as

$$\vec{E} = -\mathrm{i}\omega\left(\mathbf{I} + \frac{\nabla\nabla}{k^2}\right)\vec{A}, \quad \vec{H} = \frac{1}{\mu}\nabla\times\vec{A}, \tag{3.3}$$

where $\mathbf{I}$ is the 3×3 identity matrix, k is the wave number defined as

$$k = \sqrt{\omega^2\varepsilon\mu}. \tag{3.4}$$

From the Maxwell's equations (3.1), the constitutive relations (3.2) and the Lorenz gauge condition, one can show that $\vec{A}$ satisfies the Helmholtz equation

$$\nabla^2\vec{A} + k^2\vec{A} = \vec{0}. \tag{3.5}$$

The choice of the vector potential is *not* unique. Indeed, for any function $\phi \in C^2(\mathbb{R}^3)$ satisfying the Helmholtz equation $\nabla^2\phi + k^2\phi = 0$, we can replace $\vec{A}$ by $\vec{A} + \nabla\phi$ in (3.3) to give exactly the same $\vec{E}$ and $\vec{H}$.

The dyadic Green's functions for the free space Maxwell's equations are defined using a 3×3 potential tensor $\mathbf{G}_A^{\mathrm{f}}(\mathbf{r};\mathbf{r}')$ such that the electric field dyadic Green's function $\mathbf{G}_E(\mathbf{r};\mathbf{r}')$ and the magnetic field dyadic Green's function $\mathbf{G}_H(\mathbf{r};\mathbf{r}')$ are represented by

$$\mathbf{G}_E = -\mathrm{i}\omega\left(\mathbf{I} + \frac{\nabla\nabla}{k^2}\right)\mathbf{G}_A^{\mathrm{f}}, \quad \mathbf{G}_H = \frac{1}{\mu}\nabla\times\mathbf{G}_A^{\mathrm{f}}. \tag{3.6}$$

According to the equation (3.5) for the vector potential $\vec{A}$, the potential tensor $\mathbf{G}_A^{\mathrm{f}}$ is defined as the solution of the Helmholtz equation

$$\nabla^2 \mathbf{G}_A^{\mathrm{f}} + k^2 \mathbf{G}_A^{\mathrm{f}} = \frac{1}{\mathrm{i}\omega}\delta(\mathbf{r} - \mathbf{r}')\mathbf{I} \tag{3.7}$$

with the Silver–Mller radiation conditions [9]

$$\begin{aligned} |\hat{\mathbf{r}} \times \nabla \times \mathbf{G}_E(\mathbf{r};\mathbf{r}') - \mathrm{i}k\mathbf{G}_E(\mathbf{r};\mathbf{r}')| &= O(r^{-2}), \\ |\hat{\mathbf{r}} \times \nabla \times \mathbf{G}_H(\mathbf{r};\mathbf{r}') - \mathrm{i}k\mathbf{G}_H(\mathbf{r};\mathbf{r}')| &= O(r^{-2}), \end{aligned} \tag{3.8}$$

as $r = |\mathbf{r}| \to \infty$. Here, $\delta(\mathbf{r})$ is the Dirac Delta function and $\hat{\mathbf{r}}$ the unit direction along $\mathbf{r}$.

For the same reason mentioned above, $\mathbf{G}_A^{\mathrm{f}}$ is not unique. A commonly used solution to (3.7) is given by

$$\mathbf{G}_A^{\mathrm{f}}(\mathbf{r};\mathbf{r}') = -\frac{1}{\mathrm{i}\omega}\frac{e^{\mathrm{i}k|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}\mathbf{I} = -\frac{1}{\mathrm{i}\omega}g^{\mathrm{f}}(\mathbf{r};\mathbf{r}')\mathbf{I}, \tag{3.9}$$

where $g^{\mathrm{f}}(\mathbf{r};\mathbf{r}')$ is the free space Green's function of the Helmholtz equation. It is easy to verify that the $\mathbf{G}_E$ and $\mathbf{G}_H$ resulted from this $\mathbf{G}_A^{\mathrm{f}}$ satisfy the Silver–Mller radiation conditions (3.8).

3.2. The dyadic Green's functions of the time harmonic Maxwell's equations in layered media. Now suppose the space is horizontally stratified as layers $0, \cdots, L$ arranged from top to bottom, separated by planes $z = d_0, \cdots, z = d_{L-1}$ where $d_0 > \cdots > d_{L-1}$, and each layer is homogeneous with constant permittivity ε_j and constant permeability μ_j, $j = 0, \cdots, L$, respectively. We append the layer index to the end of the subscript of any layer-dependent variable or function to represent its value specified in that layer, e.g. the wave number in layer j is then

$$k_j = \omega\sqrt{\varepsilon_j \mu_j}, \quad j = 0, \cdots, L. \tag{3.10}$$

For simplicity, the layer index is sometimes omitted, and one may assume the variable is a piecewise constant function of z. These subscript rules are applied to the rest of this paper.

3.2.1. The equations in the spatial domain. The governing equations in the interior of each layer has the same form as in (3.1)-(3.2), while the following transmission conditions must be satisfied [9] between adjacent layers

$$[\![\mathbf{n} \times \vec{E}]\!] = \vec{0}, \quad [\![\mathbf{n} \cdot \vec{D}]\!] = 0, \quad [\![\mathbf{n} \times \vec{H}]\!] = \vec{0}, \quad [\![\mathbf{n} \cdot \vec{B}]\!] = 0. \tag{3.11}$$

$[\![\cdot]\!]$ is used to represent the jump of the value at the interface, i.e. across the interface $z = d$,

$$[\![f]\!] = \lim_{z \to d^+} f - \lim_{z \to d^-} f. \tag{3.12}$$

The dyadic Green's functions are again given by the tensor potential $\mathbf{G}_A$ as

$$\mathbf{G}_E = -\mathrm{i}\omega\left(\mathbf{I} + \frac{\nabla\nabla}{k^2}\right)\mathbf{G}_A, \quad \mathbf{G}_H = \frac{1}{\mu}\nabla \times \mathbf{G}_A. \tag{3.13}$$

The tensor potential $\mathbf{G}_A$ satisfies the Helmholtz equation

$$\nabla^2\mathbf{G}_A + k^2\mathbf{G}_A = \frac{1}{\mathrm{i}\omega}\delta(\mathbf{r}-\mathbf{r}')\mathbf{I}, \tag{3.14}$$

while the interface conditions are

$$[\![\mathbf{n}\times\mathbf{G}_E]\!] = \mathbf{0}, \quad [\![\varepsilon\mathbf{n}\cdot\mathbf{G}_E]\!] = \vec{0}^T, \quad [\![\mathbf{n}\times\mathbf{G}_H]\!] = \mathbf{0}, \quad [\![\mu\mathbf{n}\cdot\mathbf{G}_H]\!] = \vec{0}^T. \tag{3.15}$$

In horizontally layered media, $\mathbf{n} = \mathbf{e}_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$. In addition, the Green's functions $\mathbf{G}_E$ and $\mathbf{G}_H$ must satisfy upward/downward outgoing radiation conditions [17].

3.2.2. The equations in the frequency domain. Suppose $\mathbf{r} = (x, y, z)$ locates in layer t, and $\mathbf{r}' = (x', y', z')$ locates in layer j. The default layer index is set to be the index t of the target layer when not specified. We will begin with the separation of the z variable from the tensor potential $\widehat{\mathbf{G}}_A$ in the frequency domain of the 2-D Fourier transform, which will lead to a reaction field decomposition. The Fourier transform is taken from $(x-x', y-y')$ to (k_x, k_y), as was introduced in (2.1). Notations of the polar coordinate pair (k_ρ, α) are retained.

In the frequency domain, the gradient operator ∇ becomes a vector with partial derivative only on the z coordinate

$$\widehat{\nabla} = \begin{bmatrix}\mathrm{i}k_x\\ \mathrm{i}k_y\\ \partial_z\end{bmatrix}, \tag{3.16}$$

if followed by a function of z. Operators $\widehat{\nabla}\widehat{\nabla}$, $\widehat{\nabla}^2$ now become $\widehat{\nabla}\widehat{\nabla}^T$ and $\widehat{\nabla}^T\widehat{\nabla}$, respectively.

Recalling that the right-hand side of the Helmholtz equation (3.14) is nontrivial if and only if $\mathbf{r}'$ is in the same layer as $\mathbf{r}$, i.e. $j = t$, we define

$$\widehat{\mathbf{G}}^{\mathrm{r}}_A(k_x, k_y, z; z') = \widehat{\mathbf{G}}_A(k_x, k_y, z; z') - \delta_{j,t}\widehat{\mathbf{G}}^{\mathrm{f}}_A(k_x, k_y, z; z'), \tag{3.17}$$

where $\delta_{j,t}$ is the Kronecker delta function, $\widehat{\mathbf{G}}^{\mathrm{f}}_A$ is the Fourier transform of the $\mathbf{G}^{\mathrm{f}}_A$ given in (3.9). The complementary part $\widehat{\mathbf{G}}^{\mathrm{r}}_A$ is called the reaction field, and satisfies a *homogeneous* Helmholtz equation

$$\widehat{\nabla}^2\widehat{\mathbf{G}}^{\mathrm{r}}_A + k^2\widehat{\mathbf{G}}^{\mathrm{r}}_A = \mathbf{0}, \quad \text{i.e.} \quad \partial_{zz}\widehat{\mathbf{G}}^{\mathrm{r}}_A + (k^2 - k_\rho^2)\widehat{\mathbf{G}}^{\mathrm{r}}_A = \mathbf{0}. \tag{3.18}$$

Define

$$k_z = \sqrt{k^2 - k_\rho^2} \tag{3.19}$$

where the square root takes nonnegative imaginary part. The general solutions to (3.18), when treated as an ordinary differential equation of z, is given by

$$\widehat{\mathbf{G}}^{\mathrm{r}}_A = e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}^{\mathrm{r}\uparrow}_A(k_x, k_y; z') + e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}^{\mathrm{r}\downarrow}_A(k_x, k_y; z'), \tag{3.20}$$

where $\widehat{\mathbf{G}}^{\mathrm{r}\uparrow}_A$ and $\widehat{\mathbf{G}}^{\mathrm{r}\downarrow}_A$ are piecewise constants with respect to z, namely,

$$\widehat{\mathbf{G}}^{\mathrm{r}\uparrow}_A = \widehat{\mathbf{G}}^{\mathrm{r}\uparrow}_{A,t}, \quad \widehat{\mathbf{G}}^{\mathrm{r}\downarrow}_A = \widehat{\mathbf{G}}^{\mathrm{r}\downarrow}_{A,t} \tag{3.21}$$

when in layer t.

We can also write $\widehat{\mathbf{G}}^{\mathrm{f}}_A$ in a similar form

$$\widehat{\mathbf{G}}^{\mathrm{f}}_A = \frac{-1}{2\omega k_{z,j}} e^{\mathrm{i}k_{z,j}|z-z'|}\mathbf{I} = \frac{-1_{\{z>z'\}}}{2\omega k_{z,j}} e^{\mathrm{i}k_{z,j}(z-z')}\mathbf{I} + \frac{-1_{\{z<z'\}}}{2\omega k_{z,j}} e^{\mathrm{i}k_{z,j}(z'-z)}\mathbf{I} \tag{3.22}$$

when $z' \neq z$.

Hence we may alternatively use the expression for the Green's function for the vector potential

$$\widehat{\mathbf{G}}_A = e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}^{\uparrow}_A + e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}^{\downarrow}_A \tag{3.23}$$

where

$$\widehat{\mathbf{G}}^{\uparrow}_A = \delta_{j,t} 1_{\{z>z'\}} \frac{-e^{-\mathrm{i}k_{z,j}z'}}{2\omega k_{z,j}}\mathbf{I} + \widehat{\mathbf{G}}^{\mathrm{r}\uparrow}_A, \quad \widehat{\mathbf{G}}^{\downarrow}_A = \delta_{j,t} 1_{\{z<z'\}} \frac{-e^{\mathrm{i}k_{z,j}z'}}{2\omega k_{z,j}}\mathbf{I} + \widehat{\mathbf{G}}^{\mathrm{r}\downarrow}_A, \tag{3.24}$$

assuming $z \neq d_i$, $0 \leq i \leq L-1$ and $z \neq z'$.

We call the separation of variable z in (3.23) the *reaction field decomposition* of $\widehat{\mathbf{G}}_A$, since this decomposition separates the free-space part and the reaction field part, and distinguishes the wave components by propagating upwards or downwards in the vertical direction. $e^{\tau^* \mathrm{i}k_z z}\widehat{\mathbf{G}}^{*}_A$ are called the *propagation components* of $\widehat{\mathbf{G}}_A$, where

$$\tau^{\uparrow} = +1, \quad \tau^{\downarrow} = -1, \quad * \in \{\uparrow, \downarrow\}. \tag{3.25}$$

The τ^* notations are used for the rest of this paper. $e^{\tau^* \mathrm{i}k_z z}\widehat{\mathbf{G}}^{\mathrm{r}*}_A$ are called the *reaction components* of $\widehat{\mathbf{G}}_A$. Similarly for $\widehat{\mathbf{G}}_E$ and $\widehat{\mathbf{G}}_H$ we will define corresponding terms later after (3.29).

Remark 3.1. In our previous work on the Helmholtz equation in layered media [13, 15, 16], the z' variable was also separated out, so that each reaction component was further decomposed according to the "propagating direction" of z'.

Next, we will re-format the mathematical conditions of the tensor potential $\widehat{\mathbf{G}}_A$, including the interface conditions and the radiation conditions, using the matrix basis $\mathbf{J}_1, \cdots, \mathbf{J}_9$.

The dyadic Green's functions $\widehat{\mathbf{G}}_E$ and $\widehat{\mathbf{G}}_H$ in the frequency domain can be calculated via the Fourier transform of (3.13), i.e.,

$$\widehat{\mathbf{G}}_E = -\mathrm{i}\omega \left(\mathbf{I} + \frac{\widehat{\nabla}\widehat{\nabla}}{k^2} \right) \widehat{\mathbf{G}}_A, \quad \widehat{\mathbf{G}}_H = \frac{1}{\mu}\widehat{\nabla} \times \widehat{\mathbf{G}}_A. \tag{3.26}$$

Moreover, the general expression (3.23) of $\widehat{\mathbf{G}}_A$ implies that

$$\begin{aligned} \widehat{\nabla}\widehat{\mathbf{G}}_A &= \widehat{\nabla}^{+}\big(e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}^{\uparrow}_A\big) + \widehat{\nabla}^{-}\big(e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}^{\downarrow}_A\big), \\ \widehat{\nabla} \times \widehat{\mathbf{G}}_A &= \widehat{\nabla}^{+} \times \big(e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}^{\uparrow}_A\big) + \widehat{\nabla}^{-} \times \big(e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}^{\downarrow}_A\big), \end{aligned} \tag{3.27}$$

where the operators $\widehat{\nabla}^{\pm}$ are defined as

$$\widehat{\nabla}^{\pm} := \begin{bmatrix} \mathrm{i}k_x & \mathrm{i}k_y & \pm\mathrm{i}k_z \end{bmatrix}^T. \tag{3.28}$$

Thus, the Green's functions $\widehat{\mathbf{G}}_E$ and $\widehat{\mathbf{G}}_H$ can be represented as the linear combinations of reaction components of $\widehat{\mathbf{G}}_A$, with every coefficient matrices in $\mathfrak{R}^0$:

$$\begin{aligned}\widehat{\mathbf{G}}_E =& -\mathrm{i}\omega\left(\mathbf{I}+\frac{\widehat{\nabla}^+(\widehat{\nabla}^+)^T}{k^2}\right)e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\uparrow} - \mathrm{i}\omega\left(\mathbf{I}+\frac{\widehat{\nabla}^-(\widehat{\nabla}^-)^T}{k^2}\right)e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\downarrow} \\ =& -\mathrm{i}\omega\left(\mathbf{J}_1+\frac{k_\rho^2}{k^2}\mathbf{J}_2+\frac{1}{k^2}\mathbf{J}_5+\frac{\mathrm{i}k_z}{k^2}\mathbf{J}_3+\frac{\mathrm{i}k_z}{k^2}\mathbf{J}_4\right)e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\uparrow} \\ & -\mathrm{i}\omega\left(\mathbf{J}_1+\frac{k_\rho^2}{k^2}\mathbf{J}_2+\frac{1}{k^2}\mathbf{J}_5-\frac{\mathrm{i}k_z}{k^2}\mathbf{J}_3-\frac{\mathrm{i}k_z}{k^2}\mathbf{J}_4\right)e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\downarrow},\end{aligned} \tag{3.29}$$

$$\begin{aligned}\widehat{\mathbf{G}}_H =& \frac{1}{\mu}\widehat{\nabla}^+\times e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\uparrow}+\frac{1}{\mu}\widehat{\nabla}^-\times e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\downarrow} \\ =& \frac{1}{\mu}\left(\mathbf{J}_6+\mathbf{J}_7-\mathrm{i}k_z\mathbf{J}_9\right)e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\uparrow}+\frac{1}{\mu}\left(\mathbf{J}_6+\mathbf{J}_7+\mathrm{i}k_z\mathbf{J}_9\right)e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\downarrow}.\end{aligned} \tag{3.30}$$

The $\mathbf{n}\cdot$ and $\mathbf{n}\times$ operators in interface conditions (3.15), given $\mathbf{n}=\mathbf{e}_3$, are converted to their equivalent matrix forms in the frequency domain, respectively, as

$$[\![\mathbf{J}_1\widehat{\mathbf{G}}_E]\!]=\mathbf{0},\quad [\![\varepsilon\mathbf{J}_2\widehat{\mathbf{G}}_E]\!]=\mathbf{0},\quad [\![\mathbf{J}_9\widehat{\mathbf{G}}_H]\!]=\mathbf{0},\quad [\![\mu\mathbf{J}_7\widehat{\mathbf{G}}_H]\!]=\mathbf{0}. \tag{3.31}$$

Here we only check the first equations of (3.15) and (3.31) as an example. Let $\mathbf{v}=\begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}^T$ be any column of $\widehat{\mathbf{G}}_E$. The first equation in (3.15) is equivalent to the continuity equations of $\mathbf{n}\times\mathbf{v}=\mathbf{e}_3\times\mathbf{v}=\begin{bmatrix}-v_2 & v_1 & 0\end{bmatrix}^T$, while in the first equation of (3.31), $\mathbf{J}_1\mathbf{v}=\begin{bmatrix}v_1 & v_2 & 0\end{bmatrix}^T$. Both of them are equivalent to the continuity equations of v_1 and v_2.

Specifically, in the brackets of (3.31),

$$\begin{aligned}\mathbf{J}_1\widehat{\mathbf{G}}_E &= -\mathrm{i}\omega\left(\mathbf{J}_1+\frac{1}{k^2}\mathbf{J}_5+\frac{\mathrm{i}k_z}{k^2}\mathbf{J}_3\right)e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\uparrow}-\mathrm{i}\omega\left(\mathbf{J}_1+\frac{1}{k^2}\mathbf{J}_5-\frac{\mathrm{i}k_z}{k^2}\mathbf{J}_3\right)e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\downarrow}, \\ \varepsilon\mathbf{J}_2\widehat{\mathbf{G}}_E &= -\mathrm{i}\omega\varepsilon\left(\frac{k_\rho^2}{k^2}\mathbf{J}_2+\frac{\mathrm{i}k_z}{k^2}\mathbf{J}_4\right)e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\uparrow}-\mathrm{i}\omega\varepsilon\left(\frac{k_\rho^2}{k^2}\mathbf{J}_2-\frac{\mathrm{i}k_z}{k^2}\mathbf{J}_4\right)e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\downarrow}, \\ \mathbf{J}_9\widehat{\mathbf{G}}_H &= -\frac{1}{\mu}\left(\mathbf{J}_3-\mathrm{i}k_z\mathbf{J}_1\right)e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\uparrow}-\frac{1}{\mu}\left(\mathbf{J}_3+\mathrm{i}k_z\mathbf{J}_1\right)e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\downarrow}, \\ \mu\mathbf{J}_7\widehat{\mathbf{G}}_H &= \left(k_\rho^2\mathbf{J}_1+\mathbf{J}_5\right)e^{\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\uparrow}+\left(k_\rho^2\mathbf{J}_1+\mathbf{J}_5\right)e^{-\mathrm{i}k_z z}\widehat{\mathbf{G}}_A^{\downarrow}.\end{aligned} \tag{3.32}$$

The particular choice of $\mathbf{J}_9$ and $\mathbf{J}_7$ in (3.31) allows us to just use $\mathbf{J}_1,\cdots,\mathbf{J}_5$ in the above expressions. When imposed on any interface $z=d_l$, each of the equations in (3.31) is a linear equation of $\widehat{\mathbf{G}}_{A,l}^*$ and $\widehat{\mathbf{G}}_{A,l+1}^*$ with coefficients in $\mathfrak{R}^0$. Take the last one as an example. Due to (3.24) from the reaction field decomposition, in the equation $[\![\mu\mathbf{J}_7\widehat{\mathbf{G}}_H]\!]=\mathbf{0}$, the quantity on the $z\to d_l^+$ side is

$$\begin{aligned}&\left(k_\rho^2\mathbf{J}_1+\mathbf{J}_5\right)e^{\mathrm{i}k_{z,l}z}\widehat{\mathbf{G}}_{A,l}^{\uparrow}+\left(k_\rho^2\mathbf{J}_1+\mathbf{J}_5\right)e^{-\mathrm{i}k_{z,l}z}\widehat{\mathbf{G}}_{A,l}^{\downarrow} \\ =&\left(k_\rho^2\mathbf{J}_1+\mathbf{J}_5\right)\left(\widehat{\mathbf{G}}_{A,l}^{\mathrm{r}\uparrow}+\delta_{j,l}1_{\{d_l>z'\}}\frac{-e^{-\mathrm{i}k_{z,j}z'}}{2\omega k_{z,j}}(\mathbf{J}_1+\mathbf{J}_2)\right)e^{\mathrm{i}k_{z,l}z} \\ &+\left(k_\rho^2\mathbf{J}_1+\mathbf{J}_5\right)\left(\widehat{\mathbf{G}}_{A,l}^{\mathrm{r}\downarrow}+\delta_{j,l}1_{\{d_l<z'\}}\frac{-e^{\mathrm{i}k_{z,j}z'}}{2\omega k_{z,j}}(\mathbf{J}_1+\mathbf{J}_2)\right)e^{-\mathrm{i}k_{z,l}z},\end{aligned} \tag{3.33}$$

which is seen to be written using elements of $\mathfrak{R}^0$ as coefficients. The same result applies to the $z \to d_l^-$ side in layer $l+1$. So the jump equation itself at $z = d_l$ will also only involves elements of $\mathfrak{R}^0$ as coefficients.

For the upward/downward outgoing radiation conditions [17], it is sufficient to describe them in the frequency domain as the decay conditions of $\widehat{\mathbf{G}}_E$ and $\widehat{\mathbf{G}}_H$ as $z \to \pm\infty$, so that waves never *come* from $z = \pm\infty$. Such conditions are sufficient to uniquely determine the Green's function, so we don't bother to raise more complicated statements. In the top layer, by (3.29), the downwards propagation components of $\widehat{\mathbf{G}}_E$ and $\widehat{\mathbf{G}}_H$ must be zero, since its asymptotic behavior is determined by the $e^{-\mathrm{i}k_{z,0}z}$ factor, so

$$-\mathrm{i}\omega\left(\mathbf{J}_1 + \frac{k_\rho^2}{k_0^2}\mathbf{J}_2 + \frac{1}{k_0^2}\mathbf{J}_5 - \frac{\mathrm{i}k_{z,0}}{k_0^2}\mathbf{J}_3 - \frac{\mathrm{i}k_{z,0}}{k_0^2}\mathbf{J}_4\right)\widehat{\mathbf{G}}^{\downarrow}_{A,0} = \mathbf{0}, \tag{3.34}$$

$$\frac{1}{\mu_0}\left(\mathbf{J}_6 + \mathbf{J}_7 + \mathrm{i}k_{z,0}\mathbf{J}_9\right)\widehat{\mathbf{G}}^{\downarrow}_{A,0} = \mathbf{0} \tag{3.35}$$

where $\widehat{\mathbf{G}}^{\downarrow}_{A,0} = \widehat{\mathbf{G}}^{\mathrm{r}\downarrow}_{A,0}$ when $z > z'$ as it happens when $z \to \infty$. When we treat (3.34) and (3.35) as linear equations of $\widehat{\mathbf{G}}^{\mathrm{r}\downarrow}_{A,0}$, the coefficient matrix in (3.35)

$$\frac{1}{\mu_0}\left(\mathbf{J}_6 + \mathbf{J}_7 + \mathrm{i}k_{z,0}\mathbf{J}_9\right) = \frac{1}{\mu_0}\begin{bmatrix} 0 & \mathrm{i}k_{z,0} & \mathrm{i}k_y \\ -\mathrm{i}k_{z,0} & 0 & -\mathrm{i}k_x \\ -\mathrm{i}k_y & \mathrm{i}k_x & 0 \end{bmatrix} \tag{3.36}$$

clearly has rank 2, allowing one degree of freedom on each column of $\widehat{\mathbf{G}}^{\mathrm{r}\downarrow}_{A,0}$. One can verify that for arbitrary 3×1 vector $\mathbf{v}$,

$$\widehat{\mathbf{G}}^{\mathrm{r}\downarrow}_{A,0} = \widehat{\nabla}_0^- \cdot \mathbf{v}^T \tag{3.37}$$

is a solution to (3.35), thus solving the linear equation. Furthermore, we can also verify it is a solution to (3.34), so (3.35) can be neglected and keeping only (3.34) does not lose any mathematical condition. It is worth mentioning that the coefficient matrix of (3.34) is an element of $\mathfrak{R}^0$.

Similarly, as $z \to -\infty$ we get another equation in the bottom layer

$$-\mathrm{i}\omega\left(\mathbf{J}_1 + \frac{k_\rho^2}{k_L^2}\mathbf{J}_2 + \frac{1}{k_L^2}\mathbf{J}_5 + \frac{\mathrm{i}k_{z,L}}{k_L^2}\mathbf{J}_3 + \frac{\mathrm{i}k_{z,L}}{k_L^2}\mathbf{J}_4\right)\widehat{\mathbf{G}}^{\uparrow}_{A,L} = \mathbf{0}, \tag{3.38}$$

where $\widehat{\mathbf{G}}^{\uparrow}_{A,L} = \widehat{\mathbf{G}}^{\mathrm{r}\uparrow}_{A,L}$ when $z < z'$.

Now, the interface conditions (3.31) and the radiation conditions (3.34) and (3.38) together consist a linear system of the unknown tensors $\widehat{\mathbf{G}}^{\mathrm{r}*}_{A,t}$ in the reaction field for $0 \le t \le L$ and $* \in \{\uparrow, \downarrow\}$, with coefficients in $\mathfrak{R}^0$. They are in general sufficient to restrict all the $\widehat{\mathbf{G}}^{\mathrm{r}*}_{A,t}$ terms. By Theorem 2.10, there exists a solution to this linear system with each unit of the block in $\mathfrak{R}^0$, i.e. each $\widehat{\mathbf{G}}^{\mathrm{r}\uparrow}_{A}, \widehat{\mathbf{G}}^{\mathrm{r}\downarrow}_{A} \in \mathfrak{R}^0$ piecewisely in each layer. Therefore, we conclude that the reaction field decomposition for the vector potential Green's function in (3.23)

$$\widehat{\mathbf{G}}_A \in \mathfrak{R}^0 \tag{3.39}$$

has the $\mathfrak{R}^0$-matrix basis formulation.

3.2.3. Further simplification of the formulation. With the $\mathfrak{R}^0$-matrix basis formulation we can simplify the interface equations (3.31) and the radiation equations (3.34) and (3.38). Suppose $\widehat{\mathbf{G}}_A^{\mathrm{r}*} \in \mathfrak{R}^0$ has the following basis expansion

$$\widehat{\mathbf{G}}_A^{\mathrm{r}*} = \sum_{l=1}^{5} a_l^{\mathrm{r}*} \mathbf{J}_l, \quad * \in \{\uparrow, \downarrow\}. \tag{3.40}$$

Following (3.23), we define

$$a_l = \delta_{j,t} a_l^{\mathrm{f}} + e^{\mathrm{i}k_z z} a_l^{\mathrm{r}\uparrow} + e^{-\mathrm{i}k_z z} a_l^{\mathrm{r}\downarrow}, \tag{3.41}$$

where a_l^{f} are the $\mathfrak{R}^0$-matrix basis coefficients of the free space potential tensor (3.22)

$$\sum_{l=1}^{5} a_l^{\mathrm{f}} \mathbf{J}_l = \widehat{\mathbf{G}}_A^{\mathrm{f}} = \frac{-1}{2\omega k_{z,j}} e^{\mathrm{i}k_{z,j}|z-z'|} (\mathbf{J}_1 + \mathbf{J}_2). \tag{3.42}$$

Then, we have obtained the $\mathfrak{R}^0$-matrix basis expression for

$$\widehat{\mathbf{G}}_A = \sum_{l=1}^{5} a_l \mathbf{J}_l$$

using a reaction field decomposition (3.41) in each a_l. It is straightforward that each coefficient a_l satisfies a Helmholtz equation

$$\partial_{zz} a_l + k_z^2 a_l = 0 \tag{3.43}$$

piecewisely in each layer, provided $z \neq z'$.

However, the potential tensor $\widehat{\mathbf{G}}_A$ is still *not* uniquely determined provided having a $\mathfrak{R}^0$-matrix basis representation. For instance, for any functions $f_1, f_2 \in C^2(\mathbb{R}^3)$ satisfying the Helmholtz equation $\nabla^2 f_j + k^2 f_j = 0$, $j = 1, 2$, the potential tensor $\widehat{\mathbf{G}}_A + \partial_z \widehat{f}_1 \mathbf{J}_2 + \widehat{f}_1 \mathbf{J}_3 + \partial_z \widehat{f}_2 \mathbf{J}_4 + \widehat{f}_2 \mathbf{J}_5$ can be used as an alternative choice for $\widehat{\mathbf{G}}_A$. In order to eliminate these uncertainties in the coefficients, we will use less functions to represent $\widehat{\mathbf{G}}_A$ and its reaction field components instead. Define functions b_1, b_2 and b_3 by linear transforms of a_l:

$$\begin{aligned} b_1 &= a_1, \\ b_2 &= \frac{1}{\mu} \left(a_2 - \partial_z a_3 \right), \\ b_3 &= \frac{1}{\mu} \left(\partial_z a_1 + k_\rho^2 a_4 - k_\rho^2 \partial_z a_5 \right). \end{aligned} \tag{3.44}$$

Each b_l inherits the property of being the piecewise solution to the Helmholtz equation

$$\partial_{zz} b_l + k_z^2 b_l = 0, \quad z \neq z'. \tag{3.45}$$

By substituting into representations of $\widehat{\mathbf{G}}_E$ and $\widehat{\mathbf{G}}_H$ (see (3.26)) using the matrix basis coefficients, i.e.

$$\widehat{\mathbf{G}}_E = -\mathrm{i}\omega \left(\mathbf{I} + \frac{\widehat{\nabla}\widehat{\nabla}^T}{k^2} \right) \sum_{l=1}^{5} a_l \mathbf{J}_l, \quad \widehat{\mathbf{G}}_H = \frac{1}{\mu} \widehat{\nabla} \times \sum_{l=1}^{5} a_l \mathbf{J}_l,$$

we get representations of $\widehat{\mathbf{G}}_E$ and $\widehat{\mathbf{G}}_H$ using b_1, b_2 and b_3

(3.46)

$$\widehat{\mathbf{G}}_E = -\frac{\mathrm{i}\omega}{k^2}\left(k^2 b_1\mathbf{J}_1 + \mu k_\rho^2 b_2\mathbf{J}_2 + \mu\partial_z b_2\mathbf{J}_3 + \mu b_3\mathbf{J}_4 + \left(\frac{k^2}{k_\rho^2}b_1 + \frac{\mu}{k_\rho^2}\partial_z b_3\right)\mathbf{J}_5\right),$$
$$\widehat{\mathbf{G}}_H = \frac{1}{\mu}\left(b_1\mathbf{J}_6 + \mu b_2\mathbf{J}_7 + \left(\frac{1}{k_\rho^2}\partial_z b_1 - \frac{\mu}{k_\rho^2}b_3\right)\mathbf{J}_8 - \partial_z b_1\mathbf{J}_9\right),$$

i.e. using b_1, b_2 and b_3 is sufficient to represent the desired dyadic E and H Green's functions.

To derive an efficient method evaluating the b_l functions in layered media, $l = 1, 2, 3$, we begin with their reaction component decompositions. Corresponding to (3.41), we can expand each b_l as

$$b_l = b_l(k_\rho, z, z') = \delta_{j,t} b_l^{\mathrm{f}}(k_\rho, z, z') + e^{\mathrm{i}k_z z} b_l^{\mathrm{r}\uparrow}(k_\rho, z') + e^{-\mathrm{i}k_z z} b_l^{\mathrm{r}\downarrow}(k_\rho, z'), \tag{3.47}$$

where

(3.48)

$$b_1^{\mathrm{f}} = a_1^{\mathrm{f}}, \quad b_2^{\mathrm{f}} = \frac{1}{\mu}\left(a_2^{\mathrm{f}} - \partial_z a_3^{\mathrm{f}}\right), \quad b_3^{\mathrm{f}} = \frac{1}{\mu}\left(\partial_z a_1^{\mathrm{f}} + k_\rho^2 a_4^{\mathrm{f}} - k_\rho^2\partial_z a_5^{\mathrm{f}}\right),$$
$$b_1^{\mathrm{r}*} = a_1^{\mathrm{r}*}, \quad b_2^{\mathrm{r}*} = \frac{1}{\mu}\left(a_2^{\mathrm{r}*} - \tau^*\mathrm{i}k_z a_3^{\mathrm{r}*}\right), \quad b_3^{\mathrm{r}*} = \frac{1}{\mu}\left(\tau^*\mathrm{i}k_z a_1^{\mathrm{r}*} + k_\rho^2 a_4^{\mathrm{r}*} - k_\rho^2\tau^*\mathrm{i}k_z a_5^{\mathrm{r}*}\right),$$

$* \in \{\uparrow, \downarrow\}$, $\tau^\uparrow = 1$, $\tau^\downarrow = -1$. Specifically, since $\mathbf{G}_A^{\mathrm{f}} = -g^{\mathrm{f}}/(\mathrm{i}\omega)\mathbf{I}$ was chosen as the free-space component, it's clear that

$$a_1^{\mathrm{f}} = a_2^{\mathrm{f}} = -\frac{1}{\mathrm{i}\omega}\widehat{g}^{\mathrm{f}}, \quad a_3^{\mathrm{f}} = a_4^{\mathrm{f}} = a_5^{\mathrm{f}} = 0, \tag{3.49}$$

where $\widehat{g}^{\mathrm{f}} = \mathrm{i}e^{\mathrm{i}k_{z,j}|z-z'|}/(2k_{z,j})$. Therefore

$$b_1^{\mathrm{f}} = -\frac{1}{\mathrm{i}\omega}\widehat{g}^{\mathrm{f}}, \quad b_2^{\mathrm{f}} = -\frac{1}{\mathrm{i}\omega}\frac{1}{\mu_j}\widehat{g}^{\mathrm{f}}, \quad b_3^{\mathrm{f}} = -\frac{1}{\mathrm{i}\omega}\frac{1}{\mu_j}\partial_z\widehat{g}^{\mathrm{f}} = -\partial_{z'} b_2^{\mathrm{f}}. \tag{3.50}$$

In order to determine b_1, b_2 and b_3, we consider the interface equations and the radiation equations. One can easily verify the interface equations (3.15), reinterpreted in the frequency domain as in (3.31), are equivalent to the following after comparing the matrix basis coefficients. For example, using

$$\mathbf{J}_1 \cdot \widehat{\mathbf{G}}_E = -\mathrm{i}\omega\left(b_1\mathbf{J}_1 + \omega^{-2}\varepsilon^{-1}\partial_z b_2\mathbf{J}_3 + (k_\rho^{-2}b_1 + \omega^{-2}\varepsilon^{-1}k_\rho^{-2}\partial_z b_3)\mathbf{J}_5\right)$$

the continuity equations

$$[\![-\mathrm{i}\omega b_1]\!] = 0, \quad [\![-\mathrm{i}\omega^{-1}\varepsilon^{-1}\partial_z b_2]\!] = 0, \quad [\![-\mathrm{i}\omega k_\rho^{-2}b_1 - \mathrm{i}\omega^{-1}k_\rho^{-2}\varepsilon^{-1}]\!] = 0$$

are revealed. After removing constant factors from these equations, a complete list by items is given below:

$$\begin{aligned}
&[\![\mathbf{n}\times\widehat{\mathbf{G}}_E]\!] = \mathbf{0} \Leftrightarrow [\![\mathbf{J}_1\cdot\widehat{\mathbf{G}}_E]\!] = \mathbf{0} \Leftrightarrow [\![b_1]\!] = 0, \left[\!\!\left[\frac{1}{\varepsilon}\partial_z b_2\right]\!\!\right] = 0, \left[\!\!\left[\frac{1}{\varepsilon}\partial_z b_3\right]\!\!\right] = 0;\\
&[\![\varepsilon\mathbf{n}\cdot\widehat{\mathbf{G}}_E]\!] = \vec{0} \Leftrightarrow [\![\mathbf{J}_2\cdot\varepsilon\widehat{\mathbf{G}}_E]\!] = \mathbf{0} \Leftrightarrow [\![b_2]\!] = 0, [\![b_3]\!] = 0;\\
&[\![\mathbf{n}\times\widehat{\mathbf{G}}_H]\!] = \mathbf{0} \Leftrightarrow [\![\mathbf{J}_9\cdot\widehat{\mathbf{G}}_H]\!] = \mathbf{0} \Leftrightarrow [\![b_2]\!] = 0, [\![b_3]\!] = 0, \left[\!\!\left[\frac{1}{\mu}\partial_z b_1\right]\!\!\right] = 0;\\
&[\![\mu\mathbf{n}\cdot\widehat{\mathbf{G}}_H]\!] = \vec{0} \Leftrightarrow [\![\mathbf{J}_7\cdot\mu\widehat{\mathbf{G}}_H]\!] = \mathbf{0} \Leftrightarrow [\![b_1]\!] = 0.
\end{aligned} \tag{3.51}$$

The radiation equations (3.34) and (3.38) are reduced to

$$b_{l,0}^{\mathrm{r}\downarrow} = 0, \quad b_{l,L}^{\mathrm{r}\uparrow} = 0 \tag{3.52}$$

in the top and the bottom layer, respectively, i.e. waves coming from $z = \pm\infty$ are prohibited in the reaction field decomposition. The above is a total of $2L + 2$ linear equations of $b_l^{\mathrm{r}\uparrow}$ and $b_l^{\mathrm{r}\downarrow}$ from $L + 1$ layers. These linear equations are solvable, from the knowledge of the acoustic wave equation in layered media:

- $-\mathrm{i}\omega b_1$ is exactly the reflection/transmission coefficient in the frequency domain of the LMDG of the Helmholtz equation, with piecewise constant material parameters $1/\varepsilon$. Thus, we can solve b_1 in the frequency domain like solving the known scalar layered Helmholtz problem [15].
- Similarly, $-\mathrm{i}\omega\mu_j b_2$ is exactly the one with piecewise constant parameters $1/\mu$.
- The linear system regarding $b_3^{\mathrm{r}*}$ has exactly the same coefficients as $b_2^{\mathrm{r}*}$ for the unknowns, so it's solvable since $b_2^{\mathrm{r}*}$ are uniquely determined by the physical problem. Moreover,

  $$-\partial_{z'} b_2 = -\partial_{z'}\delta_{j,t} b_2^{\mathrm{f}} - e^{\mathrm{i}k_z z}\partial_{z'} b_2^{\mathrm{f}\uparrow} - e^{-\mathrm{i}k_z z}\partial_{z'} b_2^{\mathrm{f}\downarrow}, \tag{3.53}$$

  which satisfies every equation that b_3 should satisfy, so by uniqueness,

  $$b_3 = -\partial_{z'} b_2, \text{ i.e. } b_3^{\mathrm{r}*} = -\partial_{z'} b_2^{\mathrm{r}*}. \tag{3.54}$$

Remark 3.2. The b_1 and b_2 functions are corresponding to the TE mode component and the TM mode component in the E_z-H_z formulation [3, 7], respectively.

Remark 3.3. To characterize $\widehat{\mathbf{G}}_E$ and $\widehat{\mathbf{G}}_H$ we don't need the intermediate, undetermined tensor potential $\widehat{\mathbf{G}}_A$ anymore. We derived the formulation for $\widehat{\mathbf{G}}_A$ because some integral equation methods solve vector potential $\mathbf{A}$, e.g., in [5].

Remark 3.4. In some situations such as the half-space problem with the impedence boundary condition on the infinite boundary, the interface conditions are not exactly in the form of (3.15), but the result of the matrix basis formulation still holds with the same derivation.

Remark 3.5 (modes of the system). A mode of the layered media is an eigenstate without stimulation from any given source, i.e. the nontrivial solution of $\widehat{\mathbf{G}}_A^{\mathrm{r}*}$ satisfying the above interface equations and radiation equations for certain values of k_ρ, with each $\widehat{\mathbf{G}}_A^{\mathrm{f}}$ replaced by 0. It corresponds to a pole in the frequency domain [16]. In such a situation we can still derive the simplified formulation using terms b_1, b_2 and b_3, but b_3 plays an independent role and is not anymore tied with b_2, i.e. a 2-term formulation won't be sufficient.

3.2.4. The transverse potential and the Sommerfeld potential. Here we take a quick review on the transverse potential and the Sommerfeld potential formulations and show how to interpret them from the matrix basis formulation. Both formulations restrict certain 5 entries of the 3×3 tensor $\widehat{\mathbf{G}}_A$ to be nonzero, which uniquely determine the tensor potential. Here, we claim that the potential tensors in these formulations have the $\mathfrak{R}^0$-matrix basis representation, and can be derived using b_1 and b_2. Due to the uniqueness of b_1 and b_2, it suffices to explicitly construct them.

The transverse potential takes the form

$$\widehat{\mathbf{G}}_A^{\mathrm{t}} = \begin{bmatrix} \times & \times & \\ \times & \times & \\ & & \times \end{bmatrix}, \tag{3.55}$$

where each $\times$ marks a nonzero entry. We claim $\widehat{\mathbf{G}}^{\mathrm{t}}_A = a_1\mathbf{J}_1 + a_2\mathbf{J}_2 + a_5\mathbf{J}_5$. By (3.44),

$$b_1 = a_1, \quad b_2 = \frac{1}{\mu}a_2, \quad b_3 = -\partial_{z'}b_2 = \frac{1}{\mu}(\partial_z a_1 - k_\rho^2\partial_z a_5). \tag{3.56}$$

Since a_l, b_l satisfy the Helmholtz equation (3.43) and (3.45), respectively, we have

$$a_1 = b_1, \quad a_2 = \mu b_2, \quad a_5 = b_1 - \frac{\mu\partial_z\partial_{z'}b_2}{k_\rho^2k_z^2}. \tag{3.57}$$

The Sommerfeld potential takes the form

$$\widehat{\mathbf{G}}^{\mathrm{S}}_A = \begin{bmatrix} \times & & \\ & \times & \\ \times & \times & \times \end{bmatrix} \tag{3.58}$$

and we claim $\widehat{\mathbf{G}}^{\mathrm{S}}_A = a_1\mathbf{J}_1 + a_2\mathbf{J}_2 + a_4\mathbf{J}_4$. Again by (3.44),

$$b_1 = a_1, \quad b_2 = \frac{1}{\mu}a_2, \quad b_3 = -\partial_{z'}b_2 = \frac{1}{\mu}(\partial_z a_1 + k_\rho^2 a_4), \tag{3.59}$$

so

$$a_1 = b_1, \quad a_2 = \mu b_2, \quad a_4 = -\frac{\mu\partial_{z'}b_2 + \partial_z b_1}{k_\rho^2}. \tag{3.60}$$

Remark 3.6. In the transverse potential $\widehat{\mathbf{G}}^{\mathrm{t}}_A$, although the coefficient a_5 has a k_ρ^2 factor in the denominator, there's no singularity in the integrand of $\widehat{\mathbf{G}}^{\mathrm{t}}_A$ at $k_\rho = 0$ since they can be cancelled out with the entries of $\mathbf{J}_5$ (by using the (k_ρ, α) polar coordinates). The same happens to the Sommerfeld potential $\widehat{\mathbf{G}}^{\mathrm{S}}_A$, but it's not explicitly shown in the expression of $a_4\mathbf{J}_4$. Numerically, we should take some care if the values as $k_\rho \to 0$ are required.

4. Numerical validation of the Maxwell's LMDG in a 10-layer medium.

The matrix basis formulation (3.46) using coefficients $b_1(k_\rho, z, z')$, $b_2(k_\rho, z, z')$ and $b_3(k_\rho, z, z')$ in the frequency domain can be used to accurately calculate the Maxwell's LMDG.

The free-space Green's function can be computed using its closed analytical form. Meanwhile, for the reaction field, a recursive scheme to compute each reaction field component of b_1 and b_2 can be found in the appendix of [14] with minor modifications. We use (3.54) to represent b_3, while the ∂_z and $\partial_{z'}$ operators are converted to $\pm \mathrm{i}k_{z,t}$ and $\pm\mathrm{i}k_{z,j}$ factors, respectively, for each reaction field component related to its field propagation direction.

To compute the inverse Fourier transforms of (2.1) for $\widehat{\mathbf{G}}_E$ and $\widehat{\mathbf{G}}_H$, we will convert them into single integrals for each entry of the tensors. This can be done by first representing entries of the matrix basis in terms of polar coordinates (k_ρ, α), e.g.

$$-k_x^2 = k_\rho^2\left(-\frac{1}{2} - \frac{1}{4}e^{2\mathrm{i}\alpha} - \frac{1}{4}e^{-2\mathrm{i}\alpha}\right),$$

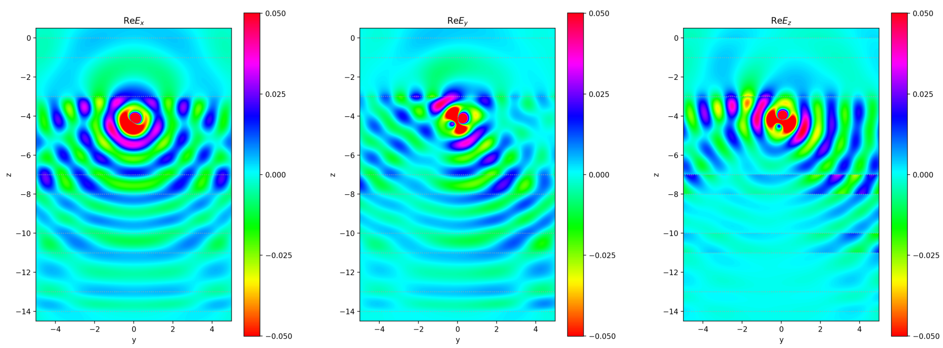
then using the identity for any integer m,

$$\begin{aligned} I_m[f] &= \iint_{\mathbb{R}^2} e^{\mathrm{i}m\alpha}e^{\mathrm{i}k_x(x-x')+\mathrm{i}k_y(y-y')}f(k_\rho)dk_xdk_y \\ &= 2\pi\mathrm{i}^m e^{\mathrm{i}m\hat{\phi}}\int_0^\infty k_\rho J_m(k_\rho\rho)f(k_\rho)dk_\rho, \end{aligned} \tag{4.1}$$

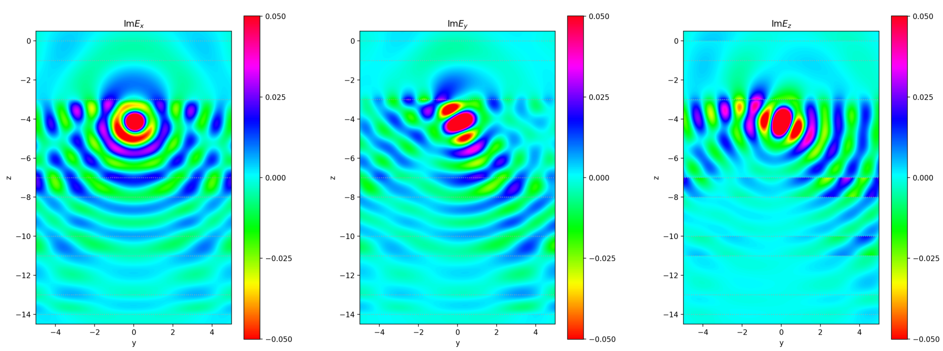
provided that the double integral is absolutely integrable. Here, $(\rho, \hat{\phi})$ is the polar coordinate pair of $(x - x', y - y')$, and the integral definition of m-th order Bessel function

$$\int_0^{2\pi} e^{\mathrm{i}k_\rho \rho \cos(\alpha - \hat{\phi})} e^{\mathrm{i}m(\alpha - \hat{\phi})} d\alpha = 2\pi \mathrm{i}^m J_m(k_\rho \rho) \tag{4.2}$$

has been used in the derivation of the identity. Each integral of the form $I_m[f]$ is computed using the Gauss–Legendre quadrature on a truncated contour in the fourth quadrant in order to get rid of singularities at wave number values.



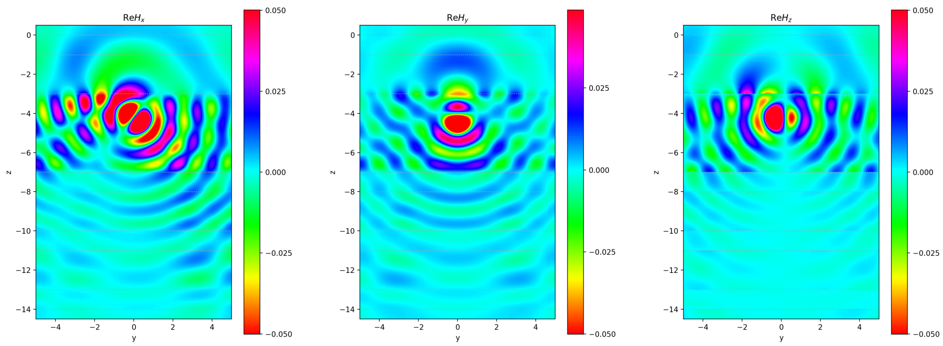
(a) $\mathrm{Re}E_x$ (b) $\mathrm{Re}E_y$ (c) $\mathrm{Re}E_z$



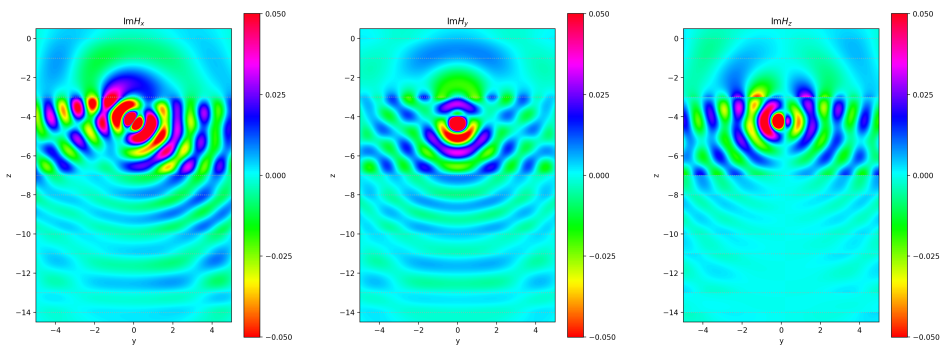
(d) $\mathrm{Im}E_x$ (e) $\mathrm{Im}E_y$ (f) $\mathrm{Im}E_z$

Fig. 4.1: Electric fields in a 10-layer medium described in Section 4. Fields are computed along $x = 0.2$ for $-5.0 \leq y \leq 5.0$ and $-14.5 \leq z \leq 0.5$. The contouring level is clamped to the range $(-0.05, 0.05)$ for a clearer illustration of wave pattern and avoiding the peak values near the source, which is located in the fourth layer at $\mathbf{r}' = (0.0, 0.0, -4.23)$ with an orientation along the direction $\hat{\boldsymbol{\alpha}}' = (1/2, 1/2, 1/\sqrt{2})$.

We consider a numerical validation test for the LMDG formula by computing the electric field Green's function of a 10-layer problem. The geometry of the layered



(a) ReH_x (b) ReH_y (c) ReH_z



(d) ImH_x (e) ImH_y (f) ImH_z

Fig. 4.2: Magnetic fields in the 10-layer medium as described in Figure 4.1.

medium is defined by horizontal interface planes $z = d_l$, $0 \leq l \leq 8$, in the descending order, as

$$\{d_l\}_{l=0}^{8} = \{0.0,\ -1.0,\ -3.0,\ -7.0,\ -8.0,\ -10.0,\ -11.0,\ -13.0,\ -14.0\}, \tag{4.3}$$

separating the space into layers with index $0, 1, \cdots, 9$ from top to bottom. Suppose layers 0–9 have constant relative permittivity

$$\{\varepsilon_l\}_{l=0}^{9} = \{1.27,\ 3.25,\ 3.41,\ 5.7,\ 1.52,\ 3.691,\ 1.2,\ 3.5,\ 2.1,\ 3.3\}, \tag{4.4}$$

respectively, and constant relative permeability

$$\{\mu_l\}_{l=0}^{9} = \{1.05,\ 0.95,\ 1.05,\ 3.95,\ 10.05,\ 6.22,\ 9.97,\ 3.2,\ 10.0,\ 1.0\}, \tag{4.5}$$

respectively. The time frequency $\omega = 1.0$. A dipole source is placed in layer no. 3 (the fourth layer) at

$$\mathbf{r}' = (0.0, 0.0, -4.23), \tag{4.6}$$

orientated along the direction

$$\hat{\boldsymbol{\alpha}}' = (1/2, 1/2, 1/\sqrt{2}). \tag{4.7}$$

Figure 4.1 shows the electric field in real and the imaginary parts of its components on the plane $x = 0.2$ for $-5.0 \le y \le 5.0$ and $-14.5 \le z \le 0.5$. Figure 4.2 shows the magnetic field in the same domain. The computed electric field and magnetic field values are clamped to the range $(-0.05, 0.05)$ for clearer illustration since the reaction field is overall smaller compared to the free-space part.

l	$e^l_{E,x}$	$e^l_{E,z}$	$e^l_{H,x}$	$e^l_{H,z}$
0	1.44e-15	1.80e-15	8.08e-16	1.57e-15
1	1.86e-15	1.14e-14	5.62e-15	9.72e-15
2	4.42e-13	3.06e-12	8.52e-13	2.14e-12
3	1.15e-13	6.33e-13	2.37e-13	5.15e-13
4	1.06e-14	2.51e-14	6.55e-15	7.01e-14
5	1.40e-15	1.05e-14	1.25e-15	1.70e-14
6	4.45e-16	2.87e-15	5.01e-16	2.13e-15
7	9.29e-17	1.34e-16	5.85e-17	4.74e-16
8	2.76e-17	1.22e-16	5.21e-17	1.86e-16

Table 4.1: Maximum absolute error of the continuity of E_x, E_y, εE_z, H_x, H_y and μH_z across interface planes $z = d_l$, $0 \le l \le 8$ for 10201 (x, y) coordinate pairs in the range of $[-5.0, 5.0] \times [-5.0, 5.0]$. The dipole source locates in layer no. 3.

The accuracy is checked by the continuity conditions at interface planes, i.e.,

$$[E_x] = 0, \quad [E_y] = 0, \quad [\varepsilon E_z] = 0, \quad [H_x] = 0, \quad [H_y] = 0, \quad [\mu H_z] = 0 \tag{4.8}$$

from (3.15). From each side of the interface plane $z = d_l$, we compute the values of E_x, E_y, εE_z, H_x, H_y and μH_z at certain (x, y) pairs within the square $[-5.0, 5.0] \times [-5.0, 5.0]$ which form a 101×101 uniform grid, then pick the maximum difference of the value jumps for each term, respectively. Namely, define

$$e^l_{E,x} = \max_{0 \le p,q \le 100} \left| E_x(x_p, y_q, d_l + 0) - E_x(x_p, y_q, d_l - 0) \right|, \tag{4.9}$$

$$e^l_{E,y} = \max_{0 \le p,q \le 100} \left| E_y(x_p, y_q, d_l + 0) - E_y(x_p, y_q, d_l - 0) \right|, \tag{4.10}$$

$$e^l_{E,z} = \max_{0 \le p,q \le 100} \left| \varepsilon_l E_z(x_p, y_q, d_l + 0) - \varepsilon_{l+1} E_z(x_p, y_q, d_l - 0) \right|, \tag{4.11}$$

$$e^l_{H,x} = \max_{0 \le p,q \le 100} \left| H_x(x_p, y_q, d_l + 0) - H_x(x_p, y_q, d_l - 0) \right|, \tag{4.12}$$

$$e^l_{H,y} = \max_{0 \le p,q \le 100} \left| H_y(x_p, y_q, d_l + 0) - H_y(x_p, y_q, d_l - 0) \right|, \tag{4.13}$$

$$e^l_{H,z} = \max_{0 \le p,q \le 100} \left| \mu_l H_z(x_p, y_q, d_l + 0) - \mu_{l+1} H_z(x_p, y_q, d_l - 0) \right|, \tag{4.14}$$

where $l = 0, 1, \cdots, 9$, $x_p = -5.0 + 0.1p$ and $y_q = -5.0 + 0.1q$, $p, q = 0, 1, \cdots, 100$. Indeed, $e^l_{E,x} = e^l_{E,y}$ and $e^l_{H,x} = e^l_{H,y}$ in this test problem because of symmetry. Table 4.1 shows the absolute errors $e^l_{E,x}$, $e^l_{E,z}$, $e_{H,x}$ and $e^l_{H,z}$ defined above, which are bounded by 3.1e-12. When only counting the interfaces from non-source layers, the absolute errors are bounded by 7.1e-14. Figure 4.1 and Figure 4.2 also indicate the continuity of computed E_x, E_y, H_x and H_y across interface planes, as well as the expected discontinuity of E_z and H_z.

5. Conclusion. In this paper, a matrix basis formulation is proposed for handling the dyadic Green's functions of the Maxwell's equations in layered media. The formulation is then used to simplify the representation and derivation of the Green's functions. In particular, the interface conditions for the electromagnetic waves is reduced to decoupled conditions for the matrix basis coefficients. As the coefficients of the matrix basis for the electric or magnetic Green's function satisfy scalar Helmholtz equations, our previously developed fast multipole method (FMM) for Helmholtz equation in 3-D layered media [13] could be extended without much technical difficulties to Maxwell's equations in 3-D layered media, which will be our future work.

Another direction of research is to apply the proposed matrix basis to investigate the dyadic Green's function of the elastic waves and analyze the S-waves and the P-waves and simplify the transmission conditions along interfaces, and furthermore to extend the FMM to elastic wave scattering in layered media.

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