Everybody's Talkin' at Me: Levels of Majority Language Acquisition by Minority Language Speakers*

William A. Brock† Bo Chen‡ Steven N. Durlauf§
Shlomo Weber¶

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Abstract

Immigrants in economies with a dominant native language exhibit substantial heterogeneities in language acquisition of the majority language. We model partial language acquisition as an equilibrium phenomenon. We consider an environment where heterogeneous agents from various minority groups choose whether to acquire a majority language fully, partially, or not at all, with varying communicative benefits and costs. We provide an equilibrium characterization of language acquisition and demonstrate that partial acquisition can arise as an equilibrium behavior. We also show that a language equilibrium may exhibit insufficient learning relative to the social optimum. In addition, we conduct a local stability analysis of steady state language equilibria. Finally, we discuss econometric implementation of the language acquisition model and establish econometric identification conditions.

Keywords: *Communicative benefits; Identification; Language acquisition; Language policy; Linguistic equilibrium; Partial learning.*

JEL classification: C72, D61, J15, Z13.

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†University of Wisconsin and University of Missouri (wbrock@ssc.wisc.edu).

‡Southern Methodist University (bochen@smu.edu).

§University of Chicago (sdurlauf@uchicago.edu).

¶New School of Economics and Southern Methodist University (sweber@smu.edu).

1 Introduction

Commonality of language has long been understood to play an essential role in promoting national solidarity while language differences can be a source of division and conflict. The distribution of language usage within a given population, therefore, matters for social stability. Further, in societies with a dominant majority language and changes, knowledge of the majority language by immigrants is an essential dimension of assimilation.

The process of majority language acquisition exhibits enormous heterogeneity across time and place. Examples of the slow convergence of language commonality abound in European contexts. Hobsbawm (1990)[41] describes how, in 1789, about half of French population did not speak French at all, and only about 12-13% spoke French well. It took more than 200 years to reach the current level of French language facility in the country, about 88% of the population. Even today, segments of the population speak various languages, as each of Breton, Corsican, German, Italian, Portuguese, Occitan, and, possibly, Picard, is used by hundreds of thousands of people.[1]

Russian/Soviet history provides another illustration of slow acquisition of the main official language by minority groups. Following the Russian-Persian war 1826-1828, the Russia Empire took control of a wide range of territories including current Georgia, Armenia and Azerbaijan. Russian became the official and administrative language in the region and was combined with systemic efforts to spread the language across the newly acquired regions. These efforts had limited success. In the early 20$^{\text{th}}$ century, it is estimated that only 3-4% of Armenians could read or speak Russian (Suny 1968[56]). The numbers increased under Soviet rule, but even then, according to the 1970 USSR Census, only 30.1% of Armenians could read or speak Russian, whereas the corresponding numbers are even lower in Azerbaijan and Georgia, 16.6% and 21.3%, respectively (Zinchenko 1972[58]).

In other cases, no common language has emerged. In Belgium, the native language of about 60% of the population is Flemish, while 40% have French as their native language. According to Eurobarometer, only 40% of Flemish-

[1] The list could be expanded to Provencal and Catalan, two dialects of Occitan.

speaking population claim to know French, whereas even a much lower number of French-speaking residents, 12%(!), speak Flemish (Ginsburgh and Weber 2011[32]). A similar disparity prevails in Canada with 75% of Anglophones and 23% of Francophones. In Canada, less than 7% of Anglophones speak French, while in Quebec, only about 35% of Francophones speak English (Statistics Canada, 2016 Census).

On the other hand, a common language can expedite the language acquisition process. In Jewish areas of Palestine, and later in Israel, Hebrew emerged as the lingua franca for various Jewish linguistic groups from North Africa, Eastern Europe, and North America who migrated to the country. It is now universally spoken among the Jewish population and widely adopted among Arab speakers. This situation in Israel can be compared to that of the United States, where it is well-known that the children of immigrants typically learn English. However, it is worth noting that English had a much slower path to becoming a common language among Native Americans.

The American case gives a different perspective on language acquisition. While de Swaan (2001)[21] has claimed that "the globalization proceeds in English," in fact globalization occurs through nonstandard English as a result of the mixing of peoples. The significance of non-native speakers' level of proficiency in English is highlighted by its inclusion in official censuses in the United States, United Kingdom, Ireland, and Australia. For instance, Table 1 displays the distributions of English-speaking skills among non-native speakers in the US, UK, Australia and Ireland, based on 2016 censuses:[2]

<table>
<tr><th>Country</th><th>Not At All</th><th>Not Well</th><th>Well</th><th>Very Well</th></tr>
<tr><td>US</td><td>5.8%</td><td>13.5%</td><td>19.8%</td><td>60.9%</td></tr>
<tr><td>UK</td><td>3.3%</td><td>17.5%</td><td colspan="2">79.2%</td></tr>
<tr><td>Australia</td><td colspan="2">16.8%</td><td colspan="2">83.2%</td></tr>
<tr><td>Ireland</td><td>2.3%</td><td>12.3%</td><td>30.1%</td><td>55.3%</td></tr>
</table>

Table 1. Distribution of English-Speaking Skills among Non-Natives

[2]The UK numbers are for 2011 and are restricted to England and Wales only. More information on partial language acquisition in censuses can be found in Section 8.1.

Table 1 reveals that a considerable proportion of non-native English speakers have only partial command of the language in each of the countries surveyed. In the US, partial learners account for approximately one third of all non-native English speakers (13.5%+19.8%). In Ireland, the fraction of non-native English speakers with a partial command of the language is even higher, at 42.4%. Table 2 demonstrates that this phenomenon of significant partial learning is also observable in countries where the dominant language is not English.[3]

	France	Germany	Spain	Italy
No Knowledge	1.7%	1.3%	5.4%	0.8%
Poorly	6.8%	8.1%	15.4%	3.9%
Just So-So	20.2%	21.6%	26.6%	12.9%
Quite Well	34.7%	41.4%	26.8%	35.1%
Almost As Well As Native	36.6%	27.7%	25.8%	47.3%

Table 2. Language Skills among Immigrants in Europe.

A persistent partial knowledge of the majority language has also been observed in Israel. A recent survey conducted among young Israelis revealed that 85% of the population in the entire country speak Hebrew very well or well, while the remaining 15% speak it poorly or not at all. Those numbers were naturally much lower for the Arab population where only 53% speak Hebrew very well or well, and 47%(!) speak it poorly or none at all.[4]

Given the stylized facts outlined above, we are prompted to examine situations in which minority language speakers choose from three levels of comprehension of the majority language: no knowledge, partial knowledge, and full knowledge. We refer to this intermediate knowledge stage as "partial learning."

We posit that the fundamental decision in language acquisition involves weighing the costs and benefits. Full language acquisition may offer more extensive communicative channels within society and potentially higher rewards

[3]See the European Internal Movers Social survey (Pioneur Project) (Alaminos et al. (2007), GESIS Data Archive, Cologne ZA4512 Data file Version 1.0.0), which provides insight into the degree of persistent partial learning of the host language by immigrants in Germany, France, Spain, Italy arrived between 1974 and 2001 (*current* rather than *upon-arrival* language data).

[4]https://www.cbs.gov.il/he/mediarelease/DocLib/2021/433/19_21_433b.pdf

than partial learning, whereas partial learning is less difficult to attain. Our work is unique in its emphasis on partial learning, as previous models have only considered language acquisition as either full or none.

A second crucial element of our approach is the consideration of heterogeneity in language acquisition among minority language speakers at both the individual and community levels. We distinguish agents via individual and group characteristics, which could, for example, be related to the level of their individual skills, their native language, and the level of literacy of their group. This allows for a discussion of the distribution of language skills in different groups in ways that can be taken to data. We consider a setting with one dominant linguistic group in the host country, and multiple immigrant or minority groups. Within each minority group, individuals differ with respect to language ability which influences the decision on the level of the dominant language to acquire.[5] We also consider an additional source of heterogeneity in population sizes of minority groups. For presentational simplicity, our model in Section 3 is offered for symmetric groups. However, we demonstrate in Section 5 that our framework can also cover and analyze the case of asymmetric population sizes.

Following the traditional theoretical literature on language acquisition (Selten and Pool (1991)[51] and Lazear (1999)[46]), we examine equilibrium outcomes in a non-cooperative language game among minority groups where the utility of minority individuals is given by their communicative benefit net of language acquisition cost. The key microfoundation of this literature is the positive dependence of the utility of every agent in the economy on the number of others with whom she can communicate with by using a common language.[6] We first address a benchmark case where all minority members face a dichotomous choice: either fully engage in the acquisition of the host language or completely refrain from learning it. This analysis extends the traditional binary approach to language acquisition in formal models.

We then expand our analysis to include the option of partial learning for

[5]Note that our model does not have intergroup complementarity/substitutability and we do not address intragroup conformity influences, as Laitin (2000)[45] does.

[6]In practice, such communicative benefits may be driven by both pure market monetary rewards and non-market benefits of access and exposure to other cultures.

minority agents. Naturally, partial learning incurs lower costs and offers lower benefits than full learning. The introduction of three ordered alternatives is a novel feature of our paper relative to existing social interactions models of discrete choice (Brock and Durlauf (2002)[11]). Our results for the three-option model differ substantially from the two-alternative setting, in terms of equilibrium behavior, comparative statics, welfare issues, network externalities, and language policy implications. Specifically, partial learning can emerge as an equilibrium choice, and the number of partial learners among minority agents can exceed that of full learners when partial learning is more valuable relative to full learning in terms of costs and benefits.

The inclusion of partial learning gives rise to some surprising results. For instance, an increase in the cost of full learning does not necessarily reduce the total number of learners, since some previous full learners may switch to partial learning, resulting in a higher number of partial learners. This peculiar linkage between costs and number of learners in equilibrium is unique to the tripartite language acquisition model and cannot be captured in a traditional binary setting. Moreover, the distinction between the dichotomous and tripartite settings leads to novel policy implications. Specifically, while subsidization of learning costs encourages language acquisition and enhances total welfare in the binary setting, it may miss a target and lead to a reduction in total welfare in the presence of partial learning. Another somewhat unexpected result is that an (explicit or implicit) ban of partial learning could reduce the welfare of both the majority and minority groups.[7] In other words, the emergence of partial learning could be welfare enhancing for the entire population.

We also study the dynamics of language learning by constructing and analyzing a deterministic dynamic process where myopic minority agents make language acquisition decisions over time. This allows us to explore the stability of the equilibria in the static version of our environment and thus speaks to likely limiting configurations of community language acquisition.

Finally, we examine how econometric analogs of the framework might be taken to data. Specifically, we discuss identification issues that arise in our

[7] See Section 8.2 in the Online Appendix for data related to the banning and restriction of partial learning in various European countries.

language framework. Here we demonstrate some interesting differences from existing results on identification of social interactions.

The paper is organized as follows. We complete this section by the review of a small but rapidly growing branch of literature on language acquisition. In Section 2, we present our model of language economy. In Section 3 we examine our equilibrium notion and analyze the effects of language policies. Section 4 is devoted to dynamics of language learning. Section 5 offers an extension of our equilibrium analysis to a setting with asymmetric minority groups. Section 6 deals with the econometric identification of the social effects that determine our equilibrium of language acquisition. And Section 7 concludes.

1.1 Literature Review on Language Acquisition

Our analysis of language acquisition builds on a small body of prior work. This prior work has exclusively focused on binary language choices: each individual either learns the other language or not and so does not address partial language acquisition. Nevertheless, important aspects of our equilibrium analysis are based on the prior literature.

In our analysis of language equilibrium, we rely on the model of Selten and Pool (1991)[51] in which the utility of every agent in the economy increases in the number of others who share a common language.[8] As we alluded to earlier, this assumption is driven by both market monetary rewards and non-market benefits from acquiring other languages. While the main objective of the Selten and Pool paper is the introduction of the equilibrium notion and the proof of its existence in a very general setting, Church and King (1993)[20] aim at characterization of linguistic equilibrium. To do this, they consider a simplified setting with two linguistic groups and homogeneous costs of language learning for all individuals in each population. Their cost homogeneity assumption produces pooling equilibria in which either the entire population acquires the other language or nobody does. To enrich the Church and King framework, Lazear (1999)[46] (see also Gabszewicz et al. (2011)[26], Ginsburgh and Weber (2011)[32]) introduces heterogeneous linguistic aptitudes, leading to the

[8] See Alcalde et al. (2022)[1] who provide an axiomatic foundation of this approach.

emergence of separating equilibria, at which a part of the population learns the other group's language, while the rest refrain from acquiring the other language. In addition to their existence and characterization results, Church and King (1993)[20], Lazear (1999)[46], and Gabszewicz et al. (2011)[26] also point out that, due to network externalities, some individuals free ride on communicative benefits generated by other members, which may lead to inefficient equilibrium levels of learning that fall below the socially optimal levels.[9]

While relying on the Selten-Pool communicative benefits model, our paper offers novel directions to the existing literature. We formally introduce the concept of partial learning, which is a widespread phenomenon where large segments of the population, especially immigrants, opt for partial rather than full command of the majority language. While partial learning was recognized by policy makers and included in population censuses, so far it has not been formally discussed in the theoretical literature. Moreover, while the papers mentioned above deal with two linguistic communities, our analysis allows for a large number of heterogeneous immigrant communities, as is the case in many countries. Another major theme of our analysis involves the dynamics of language acquisition and understanding the stability of different steady state language configurations. The closest predecessor to our work here is Marrone (2019)[50] who explores the joint evolution of knowledge of mother tongue and a dominant language in which individuals make continuous investments that determine fluency in each, and a key feature of the dynamics involves the complementarities between the stock of past investments and the marginal product of current ones. It is worth noting that our model focuses on intergroup complementarities rather than the types of complementarities in Marrone (2019)[50].

Our language acquisition model could also be linked to the social identity literature (e.g., the influential study of Shayo (2009)[53]) in that both feature externalities of individuals' choices on others in the social context. The two approaches are however not directly comparable. The social identity framework

[9] Armstrong (2015)[3] shows that this conclusion could be reversed in a model with asymmetric information where acquiring other languages is costly and the command of another language can offer a verifiable signal to employers of a bilingual employee's skill. Thus, signaling effects may outweigh network effects and the equilibrium rate of bilingualism in the economy could exceed the socially efficient level.

directly models identities as part of preferences and explicitly incorporates behaviors associated with various identities to analyze the choice and impact of societal identities. Our model, following Marschak (1965)[49], Selten and Pool (1991)[51], and Lazear (1999)[46], posits that language acquisition decisions result from a tradeoff between communicative benefits and learning costs. This allows us to derive useful economic insights, since language can be seen as purely a communicative protocol and acquiring a dominant language can be regarded as a human capital investment.

Before concluding the literature review on theoretical aspects of variegated patterns of language acquisition among various groups, we would like to point out to sociolinguistic papers that focus on the interplay of economic incentives and social factors, in particular, personal/social identity, as determinants of language acquisition (see, e.g., Joseph (2004)[43], Gumperz (2009)[38]). While such considerations are obviously important in many contexts of language acquisition, our study complements the sociolinguistic arguments by constructing a formal framework to analyze language acquisition in a multilingual society.

There is also a prior empirical literature on language acquisition. Most of this literature has focused on estimating the returns to language acquisition of foreign language by immigrants who have an incentive to learn the language of the host country if they want to assimilate with locals and find a job. These studies suggest parameter heterogeneity across environments and so provide one route by which our model can explain differences in language acquisition across contexts. Chiswick and Miller (2014)[18] identify a wide range of return values between 5% and 35%, depending on data sets, source, destination countries, languages, and gender.[10] There is also a branch of literature, albeit smaller, that examines the number of natives who acquire foreign languages to use at the workplace.[11] It turns out that acquiring a new language adds between 5 and

[10]The research for single countries covers, e.g., Australia (Chiswick and Miller 1995[17]); Canada (Aydemir and Skuterud 2005[4]); Germany (Dustmann and Van Soest 2002[24]); Israel (Beenstock et al. 2001[5]); the United Kingdom (Leslie and Lindley 2001[47]); and the US (Hellerstein and Neumark 2003[40]).

[11]For example, Canada - Shapiro and Stelcner (1997)[52], countries of the EU - Ginsburgh and Prieto-Rodriguez (2011)[31], Hungary - Galasi (2003)[27], Switzerland - Cattaneo and Winkelman (2005)[16], and the US - Fry and Lowell (2003)[25]. Interestingly, that in the

20 percent to earnings depending on the country and the language considered.

Ginsburgh et al. (2007)[30] is the rare example of a study that directly estimates language acquisition, following the Selten-Pool model. This paper derives demand functions for foreign languages estimated for English, French, German and Spanish in 13 European countries. They base their variation on three variables: the number of speakers that share this individual native language, the number of speakers of the language she considers acquiring, and the linguistic proximity between the two languages. More recently, Ginsburgh et al. (2017)[28] utilize the Selten-Pool model to estimate learning decisions by citizens in some 190 countries in the world by considering 13 of the most important world languages,[12] and identify various factors that influence individuals' learning of the language including the world population of speakers of that language and the population of speakers of that language in the country of the individuals' residence.

While we do not directly contribute to this empirical literature, we establish identification conditions for determining how language acquisition levels may be ascribed to social as opposed to individual level mechanisms.

2 A Language Economy

Consider an economy with a constant population and $(n+1)$ groups, a majority group B and n minority groups $S_i, i \in \{1, \ldots, n\}$. The population size of B is λ, and the (identical) population size of each S_i is normalized to be 1, with $\lambda > 1$.[13] Individuals in each group are initially unilingual and speak their respective native languages, denoted as $\mathfrak{b}$ for group B and $\mathfrak{s}_i$ for group S_i. Each language, $\mathfrak{b}$ or $\mathfrak{s}_i$, in the economy is assumed to be linguistically distant from another language in that communication between agents from different groups

context of Canada, Christofides and Swidinsky (2010)[19] indicated a substantial, statistically significant reward to the command of English in the French-speaking province of Quebec and insignificant effect to French in the rest of Canada.

[12] Chinese, English, Spanish, Arabic, Russian, French, Portuguese, German, Malay, Japanese, Turkish, Italian and Dutch, in descending order of number of speakers.

[13] We will consider minority groups with asymmetric population sizes in Section 5.

can only take place if the agents at least partially speak the same language.[14]

To focus on the language acquisition behavior of minority agents, we assume that majority agents do not learn any minority language, while minority agents can choose to partially or fully learn the majority language $\mathfrak{b}$ at some cost.[15] Specifically, each minority group consists of heterogeneous individuals distinguished on the basis of a linguistic cost parameter θ, i.e., the private (monetary or effort) cost of learning $\mathfrak{b}$. Minority agents with higher θ's are hence less inclined to learn $\mathfrak{b}$ than their counterparts with lower θ's. A type-θ minority agent can fully learn $\mathfrak{b}$ at cost $\ell_f\theta$, partially learn $\mathfrak{b}$ at cost $\ell_p\theta$, or choose to not learn $\mathfrak{b}$ at no cost, where $\ell_f > \ell_p > 0$.[16] Here, each language learning cost is modeled as the product of a *personal factor* (θ) and a *linguistic factor* (F, P, N), as in Selten and Pool (1991)[51]. The linguistic cost θ in each minority group is independently and identically distributed over $[0, 1]$ according to a continuously differentiable cumulative distribution function $H(\theta)$ with an everywhere positive density $h(\theta)$.

Fully or partially learning the majority language provides communicative benefits to minority agents. The communicative benefit for a minority agent is 1 if he meets someone and both of them fully know a common language. The communicative benefit is reduced to α if the minority agent partially learns $\mathfrak{b}$ and meets someone who knows $\mathfrak{b}$ fully, and further reduced to α^2 if the minority agent partially learns $\mathfrak{b}$ and meets someone who also knows $\mathfrak{b}$ partially ($0 < \alpha < 1$). To rationalize these communicative benefits, imagine that each minority agent randomly meets another in the economy to conduct a bilateral

[14]In particular, it is important that each $\mathfrak{s}_i$ is (equally) linguistically distant from $\mathfrak{b}$. However, if certain minority languages happen to be similar to each other, then additional conditions must be met. For instance, the majority group is sufficiently large ($\lambda \gg 1$) so that minority agents have no incentives to learn another minority language.

[15]This is a reasonable assumption as minority agents are more inclined to learn a majority language, which allows them access to the prevailing economic resources and opportunities. Laitin (2000)[45] however argues that minority language survival is a coordination problem and multiple languages can coexist with various language movements.

[16]For an empirical evaluation of language costs, see Carliner (2000)[15]. In addition, the importance of heterogeneity in language acquisition costs is emphasized by Bleakley and Chin (2010)[8] who use the arrival ages of immigrants to identify causal effects of English language acquisition on socioeconomic outcomes as age of arrival captures differences in language learning ability due to brain development.

trade, which can only be carried out via at least some communication between the two agents. The communicative benefits can then be interpreted as the probabilities of a successful bilateral trade, i.e., the bilateral trade takes place with probability 1 if the two parties communicate perfectly, with probability 0 if the two cannot communicate, and with probabilities α and α^2 if there is only partial communication between the two.

A minority agent hence chooses to whether fully (F), partially (P), or not (N) learn $\mathfrak{b}$. Formally, define a group strategy for minority group S_i to be $\sigma_i : [0,1] \mapsto \{F, P, N\}$, which is Borel measurable, a pure strategy of a type-θ minority agent in S_i to be $\sigma_i(\theta) \in \{F, P, N\}$, and $\sigma = (\sigma_1, ..., \sigma_n)$ to be a strategy profile for all minority agents. The (expected) payoff function of a type-θ agent in S_i given σ is:

$$u_i(\sigma;\theta) = 1 + \lambda g(\sigma_i(\theta)) + \sum\nolimits_{j\neq i} \int_0^1 g(\sigma_i(\theta))\, g(\sigma_j(t))\, dH(t) - c(\sigma_i(\theta)) \quad (1)$$

where $g(\cdot)$ denotes communicative benefits, $c(\cdot)$ is learning costs, and

$$\begin{aligned} g(\sigma_i(\theta)) &= 1 \quad \text{and } c(\sigma_i(\theta)) = \ell_f \theta \quad \text{if } \sigma_i(\theta) = F, \\ g(\sigma_i(\theta)) &= \alpha \quad \text{and } c(\sigma_i(\theta)) = \ell_p \theta \quad \text{if } \sigma_i(\theta) = P, \\ g(\sigma_i(\theta)) &= 0 \quad \text{and } c(\sigma_i(\theta)) = 0 \quad\;\;\; \text{if } \sigma_i(\theta) = N. \end{aligned}$$

Hence, the utility $u_i(\sigma;\theta)$ consists of the cost from strategy $\sigma_i(\theta)$, $c(\sigma_i(\theta))$, and the total benefit of $\sigma_i(\theta)$, which is the sum of the benefit from communicating with θ's own people in S_i (i.e., 1), the benefit from communicating with majority agents (i.e., $\lambda g(\sigma_i(\theta))$), and that from communicating with minority agents in group S_j (i.e., $\int_0^1 g(\sigma_i(\theta))\, g(\sigma_j(t))\, dH(t)$).[17] For illustration, consider $n = 2$ and the payoff of a type-θ agent in S_1 from $\sigma_1(\theta)$ is:

$$u_1(\sigma;\theta) = 1 + g(\sigma_1(\theta)) \left[\lambda + \int_0^1 g(\sigma_2(t))\, dH(t)\right] - c(\sigma_1(\theta)).$$

[17] We adopt a convenient but innocuous transformation in equation (1) where we directly use the measure of each group, rather than the probability of meeting the agents in each group, in representing the communicative benefits.

The second term in $u_1(\sigma;\theta)$ is the benefits from communicating with the majority group and the other minority group, where the integration $\int_0^1 g(\sigma_2(t))\,dH(t)$ represents the measures of agents in S_2 who choose F, P, and N.

3 Static Language Equilibrium

How will minority agents navigate their language acquisition decisions within this language economy? The payoff function in (1) highlights the importance of the trade-off between a minority agent's idiosyncratic learning cost and communicative benefits. Notably, full/partial language learning from a minority agent generates positive spillover effects for both majority agents and other minority groups, leading to strategic complementarities in the overall interaction.

We now analyze *static* language equilibria where minority agents make independent decisions on language acquisition. As all minority groups are identical, we adopt the natural solution concept of pure-strategy symmetric (Bayesian) Nash equilibrium, called **symmetric equilibrium** hereafter, where agents in different minority groups choose the same strategy if they share the same linguistic type θ:

Definition 1 *A symmetric equilibrium is a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, where $\sigma_i : [0,1] \mapsto \{F, P, N\}$ for group S_i and $\sigma_i(\theta) = \sigma_j(\theta)$ for all $i, j \in \{1, \ldots, n\}$ and $\theta \in [0,1]$, such that given $\sigma_{-i} = (\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n)$, $\sigma_i(\theta)$ is a best response for a type-θ minority agent in S_i, i.e., $u_i(\sigma_i(\theta), \sigma_{-i};\theta) \geq u_i(\sigma_i'(\theta), \sigma_{-i};\theta)$ for all $\sigma_i'(\theta)$ and θ.*

The separability and linearity of communicative benefits and learning costs suggest that minority agents adopt cutoff strategies in a symmetric equilibrium. Lemma 1 presents several key equilibrium properties:[18]

Lemma 1 *Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a symmetric equilibrium. Then*

1. *(Convexity) If types θ and θ' both choose $K \in \{F, P, N\}$, i.e., $\sigma_i^*(\theta) = \sigma_i^*(\theta') = K$ for all i, then $\sigma_i(\delta\theta + (1-\delta)\theta') = K$ for all $\delta \in (0,1)$;*

[18]Omitted proofs can be found in an Online Appendix.

2. *(Monotonicity) For any types* θ *and* θ'*, if* $\sigma_i^*(\theta) = F$ *and* $\sigma_i^*(\theta') \in \{P, N\}$*, or if* $\sigma_i^*(\theta) = P$ *and* $\sigma_i^*(\theta') = N$*, then* $\theta \leq \theta'$*;*

3. *(Positivity) There is* $\theta' > 0$ *such that* $\sigma_i^*(\theta) = F$ *for all* $\theta \in [0, \theta']$*, i.e., there is always a positive measure of full learners in* σ^**.*

Lemma 1 implies that every symmetric equilibrium is in cutoff strategies with *at most* two interior and monotonic cutoffs θ_f and θ_p, where $\theta_f < \theta_p$. Type θ_f is indifferent between full learning and partial learning, while type θ_p is indifferent between partial learning and no learning. Furthermore, Lemma 1 indicates that a symmetric equilibrium can only take one of the four following formats: (1) all minority agents fully learn $\mathfrak{b}$ (equilibrium $\mathbb{F}$), (2) all minority agents either fully or partially learn $\mathfrak{b}$ (equilibrium $\mathbb{FP}$), (3) minority agents either fully learn or not learn $\mathfrak{b}$ (equilibrium $\mathbb{FN}$), or (4) minority agents either fully learn, or partially learn, or not learn $\mathfrak{b}$ (equilibrium $\mathbb{FPN}$).

3.1 Binary Language Acquisition

To establish a baseline, we begin by examining a setting of binary language acquisition, where minority agents decide whether or not to acquire language $\mathfrak{b}$. This setting can be viewed as a special instance of our language economy where there is no incentive for partial learning ($\alpha = 0$) and/or the cost of partial learning ℓ_p is high enough to deter any minority agent from opting for partial learning in equilibrium. Our analysis here allows us to establish a connection with the existing literature, which has predominantly focused on binary language acquisition.

For equilibrium construction in this setting, consider a (common) belief that in every minority group, all types less than θ_f choose full learning (F). The payoffs from F and N for a type-θ minority agent are, respectively,

$$\begin{aligned} u(F, \theta_f; \theta) &= 1 + \lambda + (n-1) H(\theta_f) - \ell_f \theta, \\ u(N, \theta_f; \theta) &= 1. \end{aligned}$$

In a symmetric equilibrium, the equilibrium cutoff θ_f is implicitly defined as

$$\theta_f = \frac{\lambda + (n-1) H(\theta_f)}{\ell_f}. \tag{2}$$

A sufficient (but not necessary, see Figure 1) condition for the existence of at least one $\theta_f \in (0,1)$ is $\ell_f > \lambda + n - 1$. For example, if $\theta \sim \mathcal{U}[0,1]$ and $\ell_f > \lambda + n - 1$, there is a unique interior equilibrium with

$$\theta_f = \frac{\lambda}{\ell_f - (n-1)}. \tag{3}$$

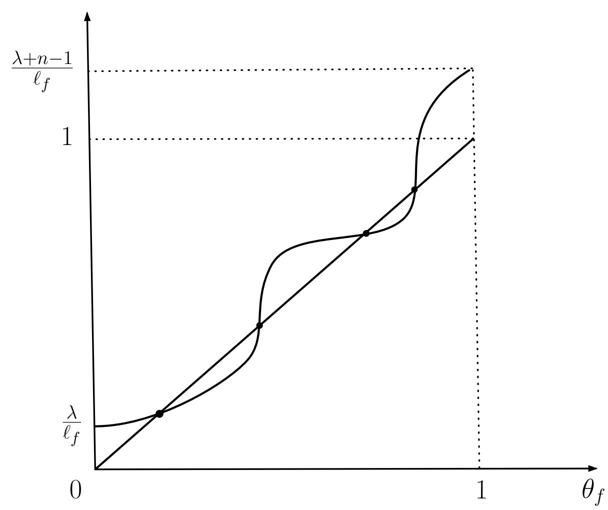


Figure 1: Multiple Equilibria in Binary Language Acquisition.

For a general distribution $H(\theta)$, however, there can be multiple (Pareto-ranked) equilibria. Figure 1 provides such an illustration where the two solid curves correspond to the RHS and LHS of equation (2), and $\ell_f < \lambda + n - 1$.

Next, we discuss the welfare implications of an (interior) equilibrium. The total welfare of an outcome where minority agents with language aptitude in $[0,\theta]$ fully acquire the majority language is, ignoring the within-group communicative benefits of $(\lambda^2 + n)$,

$$W^B(\theta) = n\left[2\lambda H(\theta) + (n-1)(H(\theta))^2 - \ell_f \int_0^\theta t dH(t)\right], \tag{4}$$

which is the difference between total communicative benefits and learning costs. For each minority group, $2\lambda H(\theta)$ is the communicative benefit with majority agents, and $(n-1)(H(\theta))^2$ is the communicative benefit with the other $(n-1)$ minority groups. A benevolent social planner chooses a language decision θ for each minority group to maximize $W^B(\theta)$.

Now consider an interior equilibrium with cutoff $\theta_f \in (0,1)$. Evaluate the derivative of $W^B(\theta)$ at $\theta = \theta_f$, using the equilibrium condition (2), to obtain

$$\begin{aligned} \left.\frac{dW^B(\theta)}{d\theta}\right|_{\theta=\theta_f} &= h(\theta_f)\, n\left[2\lambda + 2(n-1)H(\theta_f) - \ell_f\theta_f\right] \\ &= h(\theta_f)\, n\left[\lambda + (n-1)H(\theta_f)\right] > 0, \end{aligned}$$

which implies that there is **inefficient** language learning in an (interior) equilibrium. This is a familiar phenomenon in interactions with spillover effects: in the language economy, a minority agent's language acquisition generates communicative benefits for the majority group as well as the other minority groups, but such benefits are absent in the minority agent's objective, resulting in inadequate learning relative to the efficient learning outcome.

Proposition 1 summarizes our above analysis:

Proposition 1 (Language Equilibrium with Binary Acquisition) *In the language economy with binary language acquisition, a language equilibrium with cutoff θ_f is characterized by (2). Moreover, there is insufficient learning in every interior language equilibrium relative to the efficient learning outcome.*

3.2 Partial Language Acquisition

We now depart from the traditional binary-acquisition analysis and allow for partial language acquisition. We will characterize all possible equilibrium configurations. Our main objective is to examine the emergence of partial learning in equilibrium and explore the welfare implications of language equilibria.

Recall that by Lemma 1, there are four possible equilibrium configurations: equilibrium $\mathbb{F}$ and equilibrium $\mathbb{FP}$ where all minority agents fully or partially acquire language $\mathfrak{b}$, and equilibrium $\mathbb{FPN}$ and equilibrium $\mathbb{FN}$ where some minority agents choose to not learn $\mathfrak{b}$ at all. Hereafter, we characterize equilibrium conditions for each equilibrium configuration, by identifying the associated equilibrium cutoffs and incentive constraints for all types in a minority group.

Consider first equilibrium $\mathbb{FPN}$ ($\sigma^{\mathbb{FPN}}$) which is characterized by two interior cutoffs θ_f, θ_p with $0 < \theta_f < \theta_p < 1$, so that in each minority group, types in $[0, \theta_f]$ fully learn $\mathfrak{b}$, types in $(\theta_f, \theta_p]$ partially learn $\mathfrak{b}$, and types in $(\theta_p, 1]$ do not learn $\mathfrak{b}$.[19] A type-θ agent's payoffs from $\{F, P, N\}$ in an $\mathbb{FPN}$ equilibrium are

$$\begin{aligned}
u_i\left(F; \sigma^{\mathbb{FPN}}, \theta\right) &= 1 + \lambda + (n-1)\,H(\theta_f) + \alpha\,(n-1)\,[H(\theta_p) - H(\theta_f)] - \ell_f \theta, \\
u_i\left(P; \sigma^{\mathbb{FPN}}, \theta\right) &= 1 + \alpha\lambda + \alpha\,(n-1)\,H(\theta_f) + \alpha^2\,(n-1)\,[H(\theta_p) - H(\theta_f)] - \ell_p \theta, \\
u_i\left(N; \sigma^{\mathbb{FPN}}, \theta\right) &= 1.
\end{aligned}$$

The conditions for an $\mathbb{FPN}$ equilibrium can then be identified as:

$$u_i\left(F; \sigma^{\mathbb{FPN}}, \theta_f\right) = u_i\left(P; \sigma^{\mathbb{FPN}}, \theta_f\right), \tag{5}$$

$$u_i\left(P; \sigma^{\mathbb{FPN}}, \theta_p\right) = u_i\left(N; \sigma^{\mathbb{FPN}}, \theta_p\right), \tag{6}$$

$$0 < \theta_f < \theta_p < 1 \tag{7}$$

Expressions (5) and (6) characterize the equilibrium cutoffs θ_f and θ_p respectively. In addition, we can simplify (5) and (6) to obtain

$$\frac{\ell_f - \ell_p}{1 - \alpha}\theta_f = \frac{\ell_p}{\alpha}\theta_p \tag{8}$$

implying that the interior cutoffs θ_f and θ_p in an $\mathbb{FPN}$ equilibrium maintain a linear relationship regardless of the distribution $H(\cdot)$.

Next consider equilibrium $\mathbb{FP}$ ($\sigma^{\mathbb{FP}}$) which is pinned down by a single interior cutoff $\theta_f \in (0, 1)$ such that all types below θ_f fully acquire language $\mathfrak{b}$ and all types above θ_f partially acquire language $\mathfrak{b}$ in each minority group. This equilibrium arises when partial learning is sufficiently beneficial (α is large) and/or partial learning is not too costly (ℓ_p is small). We similarly write down

[19] We use the same notation θ_f, θ_p for all equilibrium formats to minimize notation.

a θ-agent's payoffs from $\{F, P\}$ as

$$\begin{aligned} u_i\left(F; \sigma^{\mathbb{FP}}, \theta\right) &= 1 + \lambda + (n-1) H(\theta_f) + \alpha (n-1) [1 - H(\theta_f)] - \ell_f \theta, \\ u_i\left(P; \sigma^{\mathbb{FP}}, \theta\right) &= 1 + \alpha\lambda + \alpha (n-1) H(\theta_f) + \alpha^2 (n-1) [1 - H(\theta_f)] - \ell_p \theta. \end{aligned}$$

The associated conditions for an $\mathbb{FP}$ equilibrium can then be written as

$$u_i\left(F; \sigma^{\mathbb{FP}}, \theta_f\right) = u_i\left(P; \sigma^{\mathbb{FP}}, \theta_f\right), \tag{9}$$

$$0 < \theta_f < 1, \tag{10}$$

$$u_i\left(P; \sigma^{\mathbb{FP}}, \theta = 1\right) \geq 1. \tag{11}$$

Here, (9) pins down the cutoff θ_f, (10) implies that the most inept type $\theta = 1$ prefers P to F, and (11) says that type $\theta = 1$ prefers P to N as well.

Equilibrium $\mathbb{FN}$ ($\sigma^{\mathbb{FN}}$), where minority agents either fully learn or not learn $\mathfrak{b}$, arises when partial learning is of little value or costly. We can calculate that

$$\begin{aligned} u_i\left(F; \sigma^{\mathbb{FN}}, \theta\right) &= 1 + \lambda + (n-1) H(\theta_f) - \ell_f \theta, \\ u_i\left(P; \sigma^{\mathbb{FN}}, \theta\right) &= 1 + \alpha\lambda + \alpha (n-1) H(\theta_f) - \ell_p \theta. \end{aligned}$$

Hence the equilibrium cutoff type θ_f coincides with (2) in the binary setting:

$$\theta_f = \frac{\lambda + (n-1) H(\theta_f)}{\ell_f}. \tag{12}$$

The conditions for equilibrium $\mathbb{FN}$ are:

$$\theta_f \in (0,1) \text{ and } u_i\left(P; \sigma^{\mathbb{FN}}, \theta_f\right) \leq u_i\left(N; \sigma^{\mathbb{FN}}, \theta_f\right) = 1. \tag{13}$$

Finally, consider equilibrium $\mathbb{F}$ ($\sigma^{\mathbb{F}}$) where all minority agents choose to fully acquire language $\mathfrak{b}$. Intuitively, this equilibrium arises whenever the cost of full learning ℓ_f is sufficiently small. For a type-θ agent in equilibrium $\mathbb{F}$, we have

$$\begin{aligned} u_i\left(F; \sigma^{\mathbb{F}}, \theta\right) &= 1 + \lambda + (n-1) - \ell_f \theta, \\ u_i\left(P; \sigma^{\mathbb{F}}, \theta\right) &= 1 + \alpha\lambda + \alpha (n-1) - \ell_p \theta. \end{aligned}$$

The incentive constraint for equilibrium $\mathbb{F}$ is hence

$$\begin{aligned} u_i\left(F;\sigma^{\mathbb{F}},1\right) &\geq \max\left\{u_i\left(P;\sigma^{\mathbb{F}},1\right), u_i\left(N;\sigma^{\mathbb{F}},1\right)\right\}, \text{ or} \\ \ell_f &\leq \min\left\{\lambda+(n-1),(1-\alpha)\lambda+(1-\alpha)(n-1)+\ell_p\right\}. \end{aligned} \tag{14}$$

As shown above, while the equilibrium cutoffs (except that for equilibrium $\mathbb{F}$) are only implicitly defined, the characterization for each equilibrium format is straightforward. In particular, the linear structure of the payoffs in our setting greatly simplifies our analysis, where the incentive constraints of all types in $[0,1]$ can be entirely reduced to some critical types' incentive constraints.[20]

Finally, we can conduct a similar welfare analysis as before (see (4)), which, together with the above equilibrium characterization, leads to:[21]

Proposition 2 (Language Equilibrium with Partial Acquisition) *In the language economy with partial acquisition, a symmetric equilibrium can be characterized by conditions* (5)-(14)*, depending on the equilibrium format. Moreover, with the exception of equilibrium* $\mathbb{F}$*, there is insufficient learning in equilibrium relative to the efficient outcome. In particular, in equilibrium* $\mathbb{FPN}$ *there is both insufficient full learning and insufficient partial learning.*

As noted previously, the binary language acquisition literature has already shown the phenomenon of insufficient language learning relative to the social optimum. Our welfare analysis in Proposition 2 identifies a similar inefficiency result, but further demonstrates that there are multiple levels of inefficiency when language acquisition can be partial. This entails different policy implications, as we will explore shortly.

3.3 Uniformly Distributed Language Aptitudes

In this section, we make the assumption that the language aptitude distribution in each minority group is uniform, i.e., $\theta \sim \mathcal{U}[0,1]$. Under this assumption,

[20]Technically, the fact that only some critical types' incentive constraints matter is due to Lemma 1, in particular, the monotonicity property of equilibria in Lemma 1.

[21]Multiple equilibria can arise in both the binary and ternary language acquisition settings. We present a numerical example in Section 9.1 (Online Appendix) to show that equilibrium multiplicity is a real, not just conceptual, phenomenon in our language economy.

we are able to provide an explicit, if technical, description of the language equilibrium. In particular, we can *trace out* the regions of parameters for all four equilibrium formats, which form a partition of the entire parameter space, indicating that there is a unique equilibrium for each parameter constellation.[22] The explicit equilibrium characterization also allows us to provide definitive answers to issues such as measures of partial learners and language policies.

We only present the analysis for equilibrium $\mathbb{FPN}$ ($\sigma^{\mathbb{FPN}}$), leaving the rest to the Online Appendix. Given two cutoffs θ_f and θ_p, the payoffs for a type-θ minority agent are

$$\begin{aligned} u_i\left(F;\sigma^{\mathbb{FPN}},\theta\right) &= 1+\lambda+(n-1)\,\theta_f+\alpha\,(n-1)\,(\theta_p-\theta_f)-\ell_f\theta, \\ u_i\left(P;\sigma^{\mathbb{FPN}},\theta\right) &= 1+\alpha\lambda+\alpha\,(n-1)\,\theta_f+\alpha^2\,(n-1)\,(\theta_p-\theta_f)-\ell_p\theta, \\ u_i\left(N;\sigma^{\mathbb{FPN}},\theta\right) &= 1. \end{aligned}$$

The indifferent types θ_f and θ_p can be explicitly calculated to be

$$\theta_f = \frac{\lambda\ell_p\,(1-\alpha)}{\ell_p\,(\ell_f-\ell_p)+(n-1)\,(2\alpha\ell_p-\alpha^2\ell_f-\ell_p)}, \tag{15}$$

$$\theta_p = \frac{\alpha\lambda\,(\ell_f-\ell_p)}{\ell_p\,(\ell_f-\ell_p)+(n-1)\,(2\alpha\ell_p-\alpha^2\ell_f-\ell_p)}. \tag{16}$$

The monotonicity property of Lemma 1 then implies that as long as θ_f and θ_p in (15) and (16) satisfy $0<\theta_f<\theta_p<1$, the incentives for all types to choose their respective equilibrium strategies are satisfied. The condition $0<\theta_f<\theta_p<1$ hence *completely* characterizes the set of parameter constellations for equilibrium $\mathbb{FPN}$.

Similar equilibrium conditions can be explicitly derived for equilibria $\mathbb{F}$, $\mathbb{FP}$, and $\mathbb{FN}$. These explicit equilibrium conditions enable us to identify the set of parameter constellations $(\ell_f,\ell_p,\lambda,\alpha,n)$ for each equilibrium format, which is summarized in Proposition 3. Given the large set of parameters involved in the characterization, we introduce two variables to help delineate the equilibrium

[22]As is standard, equilibrium uniqueness here is obtained by ignoring the equilibrium behavior of (indifferent) types with measure zero.

characterization:

$$L_f = \frac{\ell_f}{\lambda + n - 1} \text{ and } L_p = \frac{\ell_p}{\alpha\,(\lambda + n - 1)}.$$

For interpretation, L_f and L_p are respectively the 'cost and (maximum) benefit' ratios of full learning and partial learning for the extreme type $\theta = 1$. Alternatively, we can regard L_f and L_p as *relative costs* of full and partial learning respectively. As the incentives of type $\theta = 1$ are crucial for several (extreme) equilibrium formats to arise, the parameters L_f and L_p will greatly simplify our equilibrium presentation.

Proposition 3 (Language Equilibrium under Uniform Distribution) *In the language economy with uniform linguistic aptitude, there is a unique language equilibrium for each parameter constellation* $(\ell_f, \ell_p, \lambda, \alpha, n)$. *Specifically,*

[I] *if* $L_p \geq 1$, *then equilibrium* $\mathbb{F}$ *arises for* $L_f \leq 1$, *equilibrium* $\mathbb{FN}$ *arises for* $1 < L_f \leq L_p$, *and equilibrium* $\mathbb{FPN}$ *arises for* $L_p < L_f$, *while equilibrium* $\mathbb{FP}$ *does not exist;*

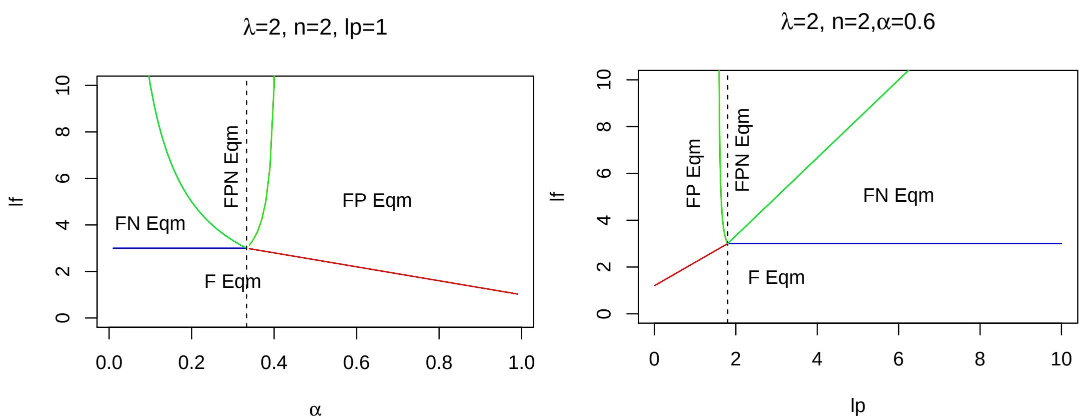
[II] *if* $L_p < 1$, *then there exist parameter thresholds* $\overline{L}_p$, $\bar{\alpha}$, *and* G *such that*[23] *equilibrium* $\mathbb{F}$ *arises for* $L_f \leq 1 - \alpha\,(1 - L_p)$, *equilibrium* $\mathbb{FPN}$ *arises for* $L_p \in \left(\overline{L}_p, 1\right)$ *and* $L_f > G$, *and equilibrium* $\mathbb{FP}$ *arises for the remaining combinations of* L_f *and* L_p, *while equilibrium* $\mathbb{FN}$ *does not exist.*

To see the intuition, first consider the case of $L_p \geq 1$, where partial learning is relatively costly. All minority agents fully learn $\mathfrak{b}$ if F is relatively inexpensive ($L_f \leq 1$). If $1 < L_f \leq L_p$, then the extreme type $\theta = 1$ prefers N to F and prefers F to P (even when all the other minority agents choose F). Hence, minority agents choose either F or N, resulting in equilibrium $\mathbb{FN}$. For a similar reason, equilibrium $\mathbb{FP}$ does not exist when $L_p \geq 1$. Finally, if $L_f > L_p$, only agents with small types choose F, while intermediate types choose P, leading to equilibrium $\mathbb{FPN}$.

[23]The explicit expressions for $\overline{L}_p$, $\bar{\alpha}$, and G, omitted in the proposition, can be found in expression (40) of the proof of Proposition 3.

The case for $L_p < 1$ where partial learning is relatively inexpensive is similar, though equilibrium analysis now is more cumbersome since one has to explicitly account for (more nuanced) trade-off between F and P. Given a small L_p, there will always be some types choosing P whenever F is not chosen by every type. Hence, equilibrium $\mathbb{FN}$ does not exist when $L_p < 1$. Next, when L_f is sufficiently small ($L_f \leq 1 - \alpha(1 - L_p)$), we similarly have that all types again fully learn $\mathbb{F}$. For larger full learning cost, i.e., $L_f > 1 - \alpha(1 - L_p)$, not all types choose F, and we then either have equilibrium $\mathbb{FPN}$ when both L_f and L_p are large, or equilibrium $\mathbb{FP}$ when either L_p or L_f is small.

Importantly, Proposition 3 shows the existence, as well as **uniqueness**, of symmetric equilibrium for each parameter constellation $(\ell_f, \ell_p, \lambda, \alpha, n)$ in the uniform setting. This is a direct implication of the fact that the characterization in Proposition 3 spans the entire space of (L_p, L_f) and the four equilibrium regions of (L_p, L_f) are *mutually exclusive*.



(a) Equilibrium Map in (α, ℓ_f)-Space (b) Equilibrium Map in (ℓ_p, ℓ_f)-Space

Figure 2: Equilibrium Maps for Proposition 3.

Proposition 3 enables us to graphically delineate the parameter constellations for all four equilibrium formats. Figure 2(a) shows a map of equilibria in the (α, ℓ_f)-space with parameters $\lambda = 2$, $n = 2$, and $\ell_p = 1$, while Figure 3(b) shows a map of equilibria in the (ℓ_p, ℓ_f)-space with $\lambda = 2$, $n = 2$, $\alpha = 0.6$.[24] In both Figures 2(a) and 2(b), the entire space is partitioned into four disjoint

[24]Explicit calculations for Figure 2(a) is in the Online Appendix, Section 9.2.

regions. The dotted vertical lines in Figure 2 (a),(b) correspond to the threshold $L_p = 1$ in Proposition 3.

The intuition behind Figure 2 is straightforward. If ℓ_f is small enough, all minority agents fully acquire the majority language, regardless of α and ℓ_p. In Figure 2(a), when $\alpha = 0$, the equilibrium outcome is consistent with that for the binary setting (see (3) and Proposition 1).[25] When ℓ_f is large, equilibrium $\mathbb{FPN}$ arises if α is intermediate, i.e., the benefit from partial learning only induces minority agents with intermediate types to partially learn. As α increases further, then even the most inept minority agents find it optimal to partially learn, resulting in equilibrium $\mathbb{FP}$. Figure 2(b) illustrates the equilibrium outcome in terms of ℓ_p, which can be interpreted similarly, except that a lower ℓ_p corresponds to a large α, making Figure 2(b) a "flipped" version of Figure 2(a).

Can partial learning be a more common choice among minority agents than full learning? We now show, using Proposition 3, that the number of partial learners can indeed exceed that of full learners among minority agents when full learning is sufficiently costly:[26]

Proposition 4 (Number of Partial Learners) *In the language equilibria $\mathbb{FP}$ and $\mathbb{FPN}$, there are strictly more partial learners than full learners in each minority group if L_f is sufficiently large.*

Hence, partial learning will be more prevalent than full learning among minority agents whenever full learning is too costly from a cost-benefit perspective. In practical terms, partial learning will be more likely to arise among minority agents if they have limited access to resources for fully acquiring it, or if partial learning meets their language needs due to limited professional opportunities.

Finally, we perform comparative statics analysis on the equilibrium measures of full learners (θ_f) and partial learners ($\theta_p - \theta_f$), focusing on equilibrium $\mathbb{FPN}$:[27]

[25] Indeed, if $\ell_f > \lambda + n - 1 = 3$, the equilibrium cutoff θ_f in (3) is interior, i.e., we have equilibrium $\mathbb{FN}$, consistent with Figure 2(a).

[26] One can explicitly show that this happens in equilibrium $\mathbb{FP}$ if $L_f > \alpha L_p + (1-\alpha)\left(1 + \frac{\lambda + \alpha(n-1)}{\lambda + n - 1}\right)$, and in equilibrium $\mathbb{FPN}$ if $L_f > (2-\alpha)\,L_p$.

[27] The role of α in our model, as illustrated in Figure 2(a),(b), is qualitatively similar to that of ℓ_p. We hence focus on α in our comparative statics analysis. Moreover, the proof of

Proposition 5 (Comparative Statics for Equilibrium $\mathbb{FPN}$) *Consider the equilibrium $\mathbb{FPN}$ in the language economy with $n \geq 1$. We have*

1. $\frac{\partial \theta_f}{\partial \lambda} > 0, \frac{\partial \theta_p}{\partial \lambda} > 0, \frac{\partial \left(\theta_p - \theta_f\right)}{\partial \lambda} > 0$ *for the measure of majority agents* (λ)*;*

2. $\frac{\partial \theta_f}{\partial n} > 0, \frac{\partial \theta_p}{\partial n} > 0, \frac{\partial \left(\theta_p - \theta_f\right)}{\partial n} > 0$ *for the number of minority groups* (n)*;*

3. $\frac{\partial \theta_f}{\partial \alpha} < 0, \frac{\partial \theta_p}{\partial \alpha} > 0, \frac{\partial \left(\theta_p - \theta_f\right)}{\partial \alpha} > 0$ *for the benefit of partial learning* (α)*;*

4. $\frac{\partial \theta_f}{\partial \ell_f} < 0, \frac{\partial \theta_p}{\partial \ell_f} \leq 0, \frac{\partial \left(\theta_p - \theta_f\right)}{\partial \ell_f} > 0$ *for the cost of full learning* (ℓ_f)*.*

Hence, a larger majority group (λ) and more minority groups (n), both strictly increasing communicative benefits from full and partial learning, give rise to more full learners and more partial learners. Likewise, a higher communicative benefit from partial learning (α) or a higher cost of full learning (ℓ_f) induces less full learners and more partial learners. Notice that changes in α and ℓ_f, which are natural candidates for policy interventions, alter the relative costs and benefits of partial and full learning, prompting minority agents to revise their learning strategies. This underscores that the presence of partial learning introduces different implications of policy interventions than those in a binary language setting, as we will discuss shortly.

Language Policies

The fact that decentralized language decisions lead to insufficient learning in our setting (Proposition 2) justifies policy interventions to facilitate minority agents' language learning. Partial learning, which arises whenever full learning is costly (large ℓ_f) or partial learning is beneficial enough (α large) by Proposition 3, induces policy implications that differ from those in the traditional binary acquisition settings. In the following, we discuss two kinds of language policies that can be employed to induce more language learning from minority agents, *pull* language policies and *push* language policies.

Proposition 5, directly using the equilibrium cutoffs θ_f and θ_p characterized in (15) and (16), is straightforward and is hence omitted.

First, consider a pull language policy where subsidies are offered to minority agents so as to encourage more language learning. To be specific, consider a policy where full learning is subsidized, perhaps due to its importance or well-established standards, so that a θ-minority agent who chooses strategy F faces a cost of $(1-\varepsilon)\ell_f\theta$ after the subsidy, where $\varepsilon \in (0,1)$ measures the intensity of the subsidy.[28] For simplicity, we focus on the case where $n = 1$ so that all minority agents are lumped into one group.[29]

How would such a subsidy policy affect the total welfare of the language economy? In an interior equilibrium of the binary language acquisition setting, the total welfare of the economy after the subsidy can be written as:[30]

$$W^B\left(\widehat{\theta}_f^\varepsilon;\varepsilon\right) = 2\lambda\widehat{\theta}_f^\varepsilon - (1-\varepsilon)\,\ell_f\int_0^{\widehat{\theta}_f^\varepsilon} tdt = \frac{2\lambda^2}{(1-\varepsilon)\,\ell_f} - \frac{\lambda^2}{2\,(1-\varepsilon)^2\,\ell_f}$$

where $\widehat{\theta}_f^\varepsilon = \frac{\lambda}{(1-\varepsilon)\ell_f}$ is the equilibrium cutoff after the subsidy. On the other hand, in an $\mathbb{FPN}$-equilibrium of our setting with partial language acquisition, the total welfare of the economy after the subsidy is

$$W^{\mathbb{FPN}}\left(\theta_f^\varepsilon,\theta_p^\varepsilon;\varepsilon\right) = 2\lambda\theta_f^\varepsilon + 2\alpha\lambda\left(\theta_p^\varepsilon - \theta_f^\varepsilon\right) - (1-\varepsilon)\,\ell_f\int_0^{\theta_f^\varepsilon} tdt - \ell_p\int_{\theta_f^\varepsilon}^{\theta_p^\varepsilon} tdt$$

where $\theta_f^\varepsilon = \frac{\lambda(1-\alpha)}{(1-\varepsilon)\ell_f - \ell_p}$ and $\theta_p^\varepsilon = \frac{\alpha\lambda}{\ell_p}$ are the cutoffs in the (subsidized) $\mathbb{FPN}$ equilibrium. Our next proposition demonstrates that the subsidy policy with a small ε, while strictly improving total welfare in the binary language setting, can surprisingly decrease total welfare *strictly* when partial learning is present:[31]

[28] For such a subsidy policy to be effective, a minority agent's full learning choice or its outcome should be verifiable. For example, a minority agent can only receive the subsidy by attending a designated language school or passing a standardized language test (e.g., TOEFL for English and JLPT for Japanese).

[29] By focusing on $n = 1$, we ignore positive spillovers of a minority agent's language acquisition on minority agents in the other minority groups. However, if the majority group is the "absolute" dominant group in the language economy (i.e., $\lambda \gg 1$), our welfare analysis below for $n = 1$ indeed provides a good approximation of welfare analysis for $n > 1$.

[30] As before, we ignore within-group communicative benefits of $\lambda^2 + n$ here.

[31] There are two reasons why we consider the case of small ε. Firstly, a small ε preserves the two equilibrium structures, which ensures the consistency of our welfare calculations. Secondly, the choice of subsidy faces various constraints in practice and a key issue is whether

Proposition 6 (Language Learning Subsidy on Total Welfare) *Consider an interior equilibrium in the binary acquisition setting and an* $\mathbb{FPN}$ *equilibrium in the setting with partial acquisition. Given a (small) subsidy intensity* $\varepsilon > 0$*,*

1. *the subsidy strictly improves total welfare in the setting with binary language acquisition, i.e.,*

$$\left.\frac{dW^B\left(\widehat{\theta}_f^\varepsilon;\varepsilon\right)}{d\varepsilon}\right|_{\varepsilon=0} > 0,$$

2. *the subsidy strictly hurts total welfare in the setting with partial language acquisition if* ℓ_p *is not too small compared to* ℓ_f*, i.e.,*

$$\left.\frac{dW^{\mathbb{FPN}}\left(\theta_f^\varepsilon,\theta_p^\varepsilon;\varepsilon\right)}{d\varepsilon}\right|_{\varepsilon=0} < 0.$$

The intuition behind Proposition 6 is as follows: In the binary setting, an increase in subsidy encourages more learning from previous non-learners, resulting in increased communicative benefits and hence total welfare in the economy. However, in the setting with partial learning, this effect is strictly smaller since the subsidy only induces full learning from previous partial learners, who were already contributing to communicative benefits (notice that θ_p^ε is independent of ε). If a significant number of previous partial learners switch to F, for example when ℓ_f and ℓ_p are close, then the (small) subsidy will actually induce **too much language learning**, which decreases total welfare. In particular, Proposition 6 shows that ignoring the existence of partial learning can lead to **misguided** language subsidy policies.

We next discuss another intuitive policy, a *push* language policy where a policy maker imposes exogenous costs, similar to a Pigouvian tax, on partial learners or non-learners in minority groups, in hopes of pushing more of them towards full learning. In the binary language acquisition setting, one can follow a similar analysis as Proposition 6 to show that a (small) additional cost on a

a subsidy should be offered at all. Our analysis exactly sheds light on this key issue.

non-learner for not learning the majority language can incentivize more minority agents to choose F, which strictly improves total welfare. In our current setting with partial learning, a policy maker can alternatively impose such a "tax" on partial learning. Our next proposition considers the extreme scenario where partial learning is **banned** from the language economy, e.g., through exorbitant administrative costs or legal/statutory requirements on language proficiency. Such a "partial learning ban" will obviously impose binding constraints (in an $\mathbb{FPN}/\mathbb{FP}$ equilibrium) and result in inferior welfare consequences to minority agents. Somewhat surprisingly, however, such a push policy will hurt the majority agents as well:

Proposition 7 (Partial Learning to Majority Welfare) *Consider a language equilibrium with $n \geq 1$ minority groups where partial learning is present (i.e., equilibria $\mathbb{FP}$ and $\mathbb{FPN}$). The majority agents are strictly worse off if partial learning is banned from the language economy.*

The rationale of Proposition 7 is that while banning partial learning indeed forces some additional minority agents to fully acquire the majority language, more previous minority partial learners switch to not learning the majority language, which results in lower communicative benefits and lower welfare for both majority agents and minority agents. Our proof of Proposition 7 also shows that banning partial learning is particularly harmful for the majority group when $n > 1$, where the existence of partial learning creates a social multiplying effect, which encourages more agents in different minority groups to fully and partially learning the dominant language.

Finally, while banning partial learning may appear too extreme or difficult to implement, policies do exist that attempt to reduce partial learning. A case in point is the degree of harshness in the language requirements for residency or citizenship. As shown by the European data in Section 8.2 (Online Appendix), harsh language exams required for residency and citizenship made partial learning difficult and in some cases almost impossible. Furthermore, discouraging partial language learning in practice can be achieved indirectly, such as through public education policies and societal norms that emphasize fully learning the majority language as the only way for assimilation.

In short, this section analyzes two settings: a dichotomous setting where individuals choose between full language acquisition or refraining from studying, and a tripartite setting where individuals have an additional option of partial learning. We provide a complete characterization of language equilibria in both settings, and demonstrate that every interior equilibrium results in insufficient learning compared to the social optimum. We also consider a case with a uniform distribution of linguistic aptitudes, which allows us to derive a unique language equilibrium and identify conditions where the number of partial learners exceeds that of full learners. Interestingly, we show that subsidizing full learning and suppressing partial learning could have adverse effects on welfare in the presence of partial learning.

4 Dynamics of Language Learning

Until now, we have analyzed a static language setting where various key parameters, especially partial and full learning costs, are fixed exogenously. However, important questions remain on how language acquisition behavior evolves over time. For example, what patterns of language acquisition behavior will prevail in the long run? Is there a tendency for all minority agents to at least partially acquire the majority language? If not, what are the factors that prevent language acquisition in the limit? To that end, we propose a dynamic framework and investigate language acquisition patterns in the long run. Specifically, we consider a deterministic language learning dynamic process where the cost of language learning decreases over time as more minority agents choose to fully/partially learn the majority language.[32]

To describe the dynamic process, first observe that Lemma 1 and the characterizations in Propositions 1-3 allow us to restrict analysis to the dynamics of the cutoff points. The dynamic process of language learning is initialized at a

[32] Grin (1992)[36] is an early analysis of minority language dynamics using a first-order linear difference equation to explore whether minority languages survive and identifies stability of a minority language related to the sensitivity of individual choices to changes in the fraction of people speaking the minority language. Our analysis considers multiple levels of acquisition from explicit microfoundations. Also see Grin (1996)[37] for a literature survey.

point in one of the four equilibrium zones (e.g., an equilibrium zone in the box diagram in the (ℓ_p, ℓ_f)-Space, Figure 2(b)). Specifically, at the initial period $(t = 0)$, the minority agents make learning decisions at the baseline learning costs ℓ_f and ℓ_p, resulting in a static language equilibrium as in Propositions 2 and 3. In period $t \geq 1$, each equilibrium cutoff pair $(\theta_{f,t}, \theta_{p,t})$ is then determined again as in Propositions 2-3 by the following "updated" cost parameters

$$\ell_{f,t} = l_f\left(\ell_f, q_{f,t-1}, \phi\right), \ \ell_{p,t} = l_p\left(\ell_p, q_{p,t-1}, \phi\right), \tag{17}$$

where the updating speed $\phi > 0$, and $q_{f,t-1}$ and $q_{p,t-1}$ are respectively the (equilibrium) fractions of agents in $[0,1]$ choosing F and P in period $t-1$, i.e.,

$$q_{f,t-1} = H\left(\theta_{f,t-1}\right), \, q_{p,t-1} = H\left(\theta_{p,t-1}\right) - H\left(\theta_{f,t-1}\right).$$

The cost parameters $\ell_{f,t}$ and $\ell_{p,t}$ are functions of the baseline costs, ℓ_f, ℓ_p, as well as the fractions of minority agents that chose F and P in $(t-1)$. Furthermore, we assume that $\ell_{f,t}, \ell_{p,t}$ decrease as $q_{f,t-1}, q_{p,t-1}$ increase and $\ell_{f,t} \to \ell_f$ and $\ell_{p,t} \to \ell_p$ as $\phi \to 0$. The parameter ϕ determines the speed at which the language learning costs are updated based on $q_{f,t-1}$ and $q_{p,t-1}$ in each period. For technical and expositional convenience, we restrict attention to the uniform setting and consider the following explicit cost parameter functions:

$$\ell_{f,t} = \ell_f e^{-\phi q_{f,t-1}}, \, q_{f,t-1} \in [0,1]\,; \ \ell_{p,t} = \ell_p e^{-\phi q_{p,t-1}}, \, q_{p,t-1} \in [0,1]\,. \tag{18}$$

To summarize, we analyze a sequence of language economies where minority agents make *myopic* language decisions in each period, based on updated cost parameters $\ell_{f,t}$, $\ell_{p,t}$ and *rational expectations* that all minority agents have full structural understanding of the economy and make decisions according to the static equilibrium in period t. The interpretation of the cost functions in (18) is that the acquisition outcomes in period $t-1$ (i.e., $q_{f,t-1}$ and $q_{p,t-1}$) affect the language learning costs in period t (i.e., $\ell_{f,t}$ and $\ell_{p,t}$) in that the more minority agents acquire language $\mathfrak{b}$ in period $t-1$, the more experienced these agents are so that they learn $\mathfrak{b}$ at lower costs in period t.[33]

Remark 1 *In our dynamic process, minority agents "forget" the acquired language and make myopic language decisions in the next period. This assumption, while not intuitive, is a simplifying assumption. Alternatively, if only a proportion of minority agents forget the acquired language (i.e., partial depreciation), a similar local stability result can be obtained with a sufficiently small learning speed* ϕ. *Another alternative is to introduce a state variable of "language capital" to capture language learning status, and analyze the myopic agents' language decisions over time based on the language capital in each period. In this alternative dynamic model, while the analysis will be different, it is possible to show that stability properties of static language equilibria again hinges on the learning speed* ϕ *being sufficiently small.*[34]

The dynamics model described above is specific and the associated learning mechanism is certainly not exhaustive in capturing all possible scenarios that can be considered here. However, this dynamic analysis offers a valuable perspective for understanding how dynamic analogies of the static model may evolve. More importantly, the dynamic model enables us to investigate (local) stability properties of the static language equilibrium in Section 3. In the remainder of this section, we will study the trajectory and limiting behavior of the above dynamic process initialized at a point in one of the four equilibrium zones identified in Proposition 3. As the dynamic process initiated from the interior of the equilibrium-$\mathbb{F}$ zone halts and remains at the initial point indefinitely, we will

[33] For simplicity, we assumed in (17) that minority agents update their learning costs based only on the last-period's learning outcome $(q_{f,t-1}, q_{p,t-1})$. However, in a more general case, one can consider scenarios where minority agents accumulate "language capital" over time, resulting in $\ell_{f,t}$ and $\ell_{p,t}$ depending on $q_{f,s}, q_{p,s}$ for all $s \leq t-1$. The case in (17) can be seen as one where all accumulated language capital before $t-1$ is *fully depreciated* in t.

[34] To be more explicit on the alternative model, let the learning costs in period $t \geq 0$ be $\ell_{f,t} = \ell_f \kappa_{f,t}^{-\phi}$ and $\ell_p \kappa_{p,t}^{-\phi}$ where ϕ is again the learning speed and the full and partial language **capital stocks** are defined respectively as $\kappa_{f,t} = (1-d)\,\kappa_{f,t-1} + A_f$ and $\kappa_{p,t} = (1-d)\,\kappa_{p,t-1} + A_p$ with $\kappa_{f,0} = \kappa_{p,0} = 0$, $d \in [0,1)$ the depreciation rate and positive constants $A_p > A_f$. Minority agents again make myopic (equilibrium) language decisions in each period. For example, when $n=1$, one can verify that if we start from a static interior $\mathbb{FPN}$-equilibrium, the dynamic process will remain and converge to a nearby $\mathbb{FPN}$ equilibrium if $\ell_f > \ell_p \geq 1$, A_p is sufficiently greater than A_f, and ϕ is sufficiently small.

henceforth examine cases where the initial point of the dynamics corresponds to an $\mathbb{FN}$ equilibrium, an $\mathbb{FP}$ equilibrium, or an $\mathbb{FPN}$ equilibrium. The learning dynamics of an $\mathbb{FP}$ equilibrium exhibit qualitative similarity to that of an $\mathbb{FN}$ equilibrium, and are hence relegated to the Online Appendix (Section 9.3).

4.1 Language Learning Dynamics: F vs N

We begin with the case where the initial point is in the interior of the $\mathbb{FN}$-equilibrium zone. We proceed hereafter as if we were in the baseline setting of binary language acquisition, where minority agents are limited to choosing either F or N. As such, our analysis here directly provides dynamic stability results for the equilibrium in the traditional binary acquisition literature. Towards the end of this section, we will discuss how these stability results also establish dynamic stability of a learning dynamics initiated at an equilibrium $\mathbb{FN}$ for our model with partial learning.

In this binary setting, with the given $(t-1)$-equilibrium cutoff $\theta_{f,t-1}$ and the rational expectation that minority agents with types less than $\theta_{f,t}$ fully acquire language $\mathfrak{b}$ in period t, the payoff from F for a type-θ minority agent is

$$u^t\left(F,\theta_{f,t-1};\theta\right) = 1+\lambda+(n-1)\,\theta_{f,t} - \ell_f e^{-\phi q_{f,t-1}}\theta,$$

where the learning cost of F is due to (18) and $q_{f,t-1} = \theta_{f,t-1}$.

The equilibrium cutoff, $\theta_{f,t}$, in each minority group in t, is determined as:

$$1+\lambda+(n-1)\,\theta_{f,t} - \ell_f e^{-\phi\theta_{f,t-1}}\theta_{f,t} = 1.$$

We define a dynamics "driver" function $r(\cdot)$ as:

$$\theta_{f,t} = \frac{\lambda}{\ell_f e^{-\phi\theta_{f,t-1}} - (n-1)} \equiv r\left(\theta_{f,t-1}\right), \tag{19}$$

with $r(0) = \lambda/\left(\ell_f - n + 1\right)$ and $r(1) = \lambda/\left(\ell_f e^{-\phi} - n + 1\right)$. We impose the following assumption:

Assumption 1 $r(0) > 0$ *and* $r(1) < 1$*, or* $\ell_f e^{-\phi} > \lambda + n - 1$.

It is immediate to verify that under Assumption 1, the function $r(\cdot)$ is positive, strictly increasing, and strictly convex on $[0,1]$.

A steady state of the binary learning dynamics is defined as a language acquisition outcome $\theta^* \in [0,1]$ such that

$$\theta^* = r(\theta^*) = \frac{\lambda}{\ell_f e^{-\phi\theta^*} - (n-1)}, \tag{20}$$

which is obtained when $r(\theta)$ intersects the 45-degree line in the $(\theta, r(\theta))$ space.

Since the dynamics driver function $r(\cdot)$ is strictly increasing and strictly convex, there is a unique (interior) steady state θ^* with $\theta^* = r(\theta^*)$ and

$$\left.\frac{dr(\theta)}{d\theta}\right|_{\theta=\theta^*} = \frac{\lambda\phi\ell_f e^{-\theta^*\phi}}{\left(\ell_f e^{-\theta^*\phi} - n + 1\right)^2} < 1.$$

As a result, the steady state is *globally stable* from any initial condition.

Proposition 8 summarizes the above discussion:[35]

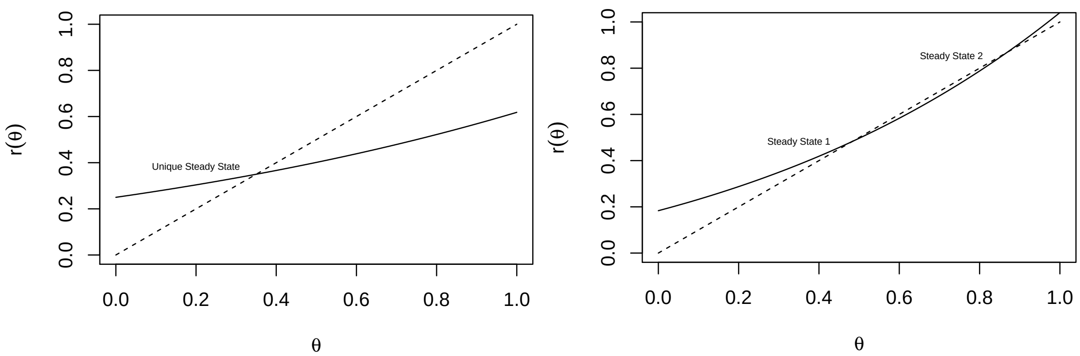
Proposition 8 *Suppose Assumption 1 holds. In the binary language learning dynamics between F and N, there is a unique steady state $\theta^* = r(\theta^*)$ with $\theta^* \in (0,1)$. In addition, the unique steady state is stable.*

Figure 3 illustrates Proposition 8. Figure 3(a) presents a unique steady state as the intersection of the dotted 45-degree line and the solid curve $r(\theta)$ (with parameters $\lambda = 2$, $n = 2$, $\ell_f = 8$, and $\phi = 0.5$), while Figure 3(b) illustrates a scenario with two steady states (with parameters $\lambda = 1.1$, $n = 3$, $\ell_f = 6$, and $\phi = 0.7$). In particular, Figure 3(b) demonstrates that when the learning speed ϕ is sufficiently large, Assumption 1 is violated, resulting in two steady states, one stable (steady state 1) and the other unstable (steady state 2).

Observe that Assumption 1 is closely related to the condition for an interior equilibrium in the static setting. Indeed, when $\phi = 0$, Assumption 1 reduces to the interior equilibrium condition for the binary acquisition setting (see (3)). This implies that the globally stable steady state θ^* is "close" to the interior equilibrium cutoff in the static binary acquisition setting when the learning speed

[35] Using stability and the implicit function theorem, one can derive some anticipated comparative statics results for the unique steady state θ^*, i.e., $\frac{\partial\theta^*}{\partial\lambda} > 0$, $\frac{\partial\theta^*}{\partial n} > 0$, $\frac{\partial\theta^*}{\partial\phi} > 0$, $\frac{\partial\theta^*}{\partial\ell_f} < 0$.

ϕ is sufficiently close to 0. Thus, our dynamic analysis here provides dynamic justification for the interior equilibrium commonly studied in the traditional binary language acquisition literature.



(a) Unique Steady State (b) Multiple Steady States

Figure 3: Binary Language Learning Dynamics: F vs N.

Finally, since the traditional binary acquisition setting can be regarded as a special case in our general acquisition setting where either α is small or ℓ_p is sufficiently large, our stability results in this section also imply that if the dynamic process starts at an (interior) $\mathbb{FN}$ equilibrium, the dynamics will remain in the $\mathbb{FN}$-equilibrium zone as long as ϕ is sufficiently small, thereby establishing a local stability result for an interior equilibrium in the $\mathbb{FN}$-equilibrium zone.[36]

4.2 Language Learning Dynamics: F vs P vs N

We now move to a dynamic analysis for the $\mathbb{FPN}$-equilibrium zone. We start from an initial $\mathbb{FPN}$ equilibrium $(\theta_{f,0}, \theta_{p,0})$ with $0 < \theta_{f,0} < \theta_{p,0} < 1$ at $t = 0$ and analyze (local) stability of the initial equilibrium $(\theta_{f,0}, \theta_{p,0})$, i.e., whether the stable steady state of our dynamics comes close to the initial equilibrium point when ϕ is sufficiently small. Hereafter, we only present key steps in our analysis, given our modest objective (local stability) and the smoothness of the dynamic

[36] Assumption 1 implies a unique interior and globally stable steady state θ^* when minority agents only choose from $\{F, N\}$. However, a sufficiently large ϕ can violate Assumption 1, resulting in a non-interior steady state. In the setting where minority agents can choose from $\{F, P, N\}$, a large ϕ can drive the dynamics to escape the zone of equilibrium $\mathbb{FN}$.

system. A more detailed and precise analysis of the dynamics in the $\mathbb{FPN}$-equilibrium zone can be found in the Appendix (Section 9.4) where we specialize to a specific setting of the dynamics to gain a more precise understanding of the factors influencing local stability.

Given the $(t-1)$-equilibrium cutoffs $(\theta_{f,t-1}, \theta_{p,t-1})$ and the expectation of the equilibrium cutoffs $\theta_{f,t}$ and $\theta_{p,t}$ in period t, the payoffs from F and P in period t for a type-θ minority agent are respectively:

$$
\begin{aligned}
u^t\left(F, \theta_{f,t-1}, \theta_{p,t-1}; \theta\right) &= 1 + \lambda + (n-1)\,\theta_{f,t} + \alpha\,(n-1)\left(\theta_{p,t} - \theta_{f,t}\right) - \ell_f e^{-\phi q_{f,t-1}}\theta, \\
u^t\left(P, \theta_{f,t-1}, \theta_{p,t-1}; \theta\right) &= 1 + \alpha\lambda + \alpha\,(n-1)\,\theta_{f,t} + \alpha^2\,(n-1)\left(\theta_{p,t} - \theta_{f,t}\right) - \ell_p e^{-\phi q_{p,t-1}}\theta.
\end{aligned}
$$

Here, observe that all the nonlinearity is in the cost functions.

Using expressions (15) and (16), we derive a linear dynamic system:

$$
\begin{aligned}
\theta_{f,t} &= \lambda \ell_{p,t}\,(1-\alpha)\,/D_t, \theta_{p,t} = \alpha\lambda\left(\ell_{f,t} - \ell_{p,t}\right)/D_t \\
D_t &= \ell_{p,t}\left(\ell_{f,t} - \ell_{p,t}\right) + (n-1)\left(2\alpha\ell_{p,t} - \alpha^2\ell_{f,t} - \ell_{p,t}\right), \text{where} \\
\ell_{p,t} &= \ell_p e^{-\phi q_{p,t-1}} = \ell_p e^{-\phi\left(\theta_{p,t-1} - \theta_{f,t-1}\right)}, \ell_{f,t} = \ell_f e^{-\phi q_{f,t-1}} = \ell_f e^{-\phi\theta_{f,t-1}}.
\end{aligned} \tag{21}
$$

A steady state of the above learning dynamics starting from an $\mathbb{FPN}$ equilibrium is defined as an acquisition outcome $(\theta_f^*, \theta_p^*) \in [0,1]^2$ such that

$$
\theta_f^* = \frac{\lambda \ell_p e^{-\phi(\theta_p^* - \theta_f^*)}\,(1-\alpha)}{\left\{\begin{array}{c} \ell_p e^{-\phi(\theta_p^* - \theta_f^*)}\left(\ell_f e^{-\phi\theta_f^*} - \ell_p e^{-\phi(\theta_p^* - \theta_f^*)}\right) + \\ (n-1)\left(2\alpha\ell_p e^{-\phi(\theta_p^* - \theta_f^*)} - \alpha^2 \ell_f e^{-\phi\theta_f^*} - \ell_p e^{-\phi(\theta_p^* - \theta_f^*)}\right) \end{array}\right\}}, \tag{22}
$$

$$
\theta_p^* = \frac{a\lambda\left(\ell_f e^{-\phi\theta_f^*} - \ell_p e^{-\phi(\theta_p^* - \theta_f^*)}\right)}{\left\{\begin{array}{c} \ell_p e^{-\phi(\theta_p^* - \theta_f^*)}\left(\ell_f e^{-\phi\theta_f^*} - \ell_p e^{-\phi(\theta_p^* - \theta_f^*)}\right) + \\ (n-1)\left(2\alpha\ell_p e^{-\phi(\theta_p^* - \theta_f^*)} - \alpha^2 \ell_f e^{-\phi\theta_f^*} - \ell_p e^{-\phi(\theta_p^* - \theta_f^*)}\right) \end{array}\right\}}. \tag{23}
$$

We write the learning dynamics (21) in a useful matrix form

$$
\begin{pmatrix} \theta_{f,t} \\ \theta_{p,t} \end{pmatrix} = \begin{pmatrix} g_f\left(\ell\left(\theta_{t-1}, \phi\right), \mathbf{a}\right) \\ g_p\left(\ell\left(\theta_{t-1}, \phi\right), \mathbf{a}\right) \end{pmatrix} = \begin{pmatrix} \lambda \ell_{p,t}\,(1-\alpha)\,/D_t \\ \alpha\lambda\left(\ell_{f,t} - \ell_{p,t}\right)/D_t \end{pmatrix},
$$

where $\theta_{t-1} = (\theta_{f,t-1}, \theta_{p,t-1})$, $\ell(\theta_{t-1}, \phi) = \left(\ell_p e^{-\phi(\theta_{p,t-1}-\theta_{f,t-1})}, \ell_f e^{-\phi\theta_{f,t-1}}\right)$, and $\mathbf{a} = (\alpha, n, \lambda)$. The expressions (22) and (23) for a steady state can then be written as ($\theta^* = (\theta_f^*, \theta_p^*)$):

$$\begin{pmatrix} \theta_f^* \\ \theta_p^* \end{pmatrix} = \begin{pmatrix} g_f(\ell(\theta^*, \phi), \mathbf{a}) \\ g_p(\ell(\theta^*, \phi), \mathbf{a}) \end{pmatrix}. \tag{24}$$

To investigate local stability of the steady state defined in (24), let $\theta'_{f,t}$, $\theta'_{f,t-1}$, $\theta'_{p,t}$, $\theta'_{p,t-1}$ be small departures from (θ_f^*, θ_p^*). We then have

$$\begin{pmatrix} \theta'_{f,t} \\ \theta'_{p,t} \end{pmatrix} = \begin{pmatrix} \frac{\partial g_f(\ell(\theta^*,\phi),\mathbf{a})}{\partial \theta_f} & \frac{\partial g_f(\ell(\theta^*,\phi),\mathbf{a})}{\partial \theta_p} \\ \frac{\partial g_p(\ell(\theta^*,\phi),\mathbf{a})}{\partial \theta_f} & \frac{\partial g_p(\ell(\theta^*,\phi),\mathbf{a})}{\partial \theta_p} \end{pmatrix} \begin{pmatrix} \theta'_{f,t-1} \\ \theta'_{p,t-1} \end{pmatrix}, \tag{25}$$

with initial point $(\theta'_{f,0}, \theta'_{p,0})$ given and $t \in \mathbb{N}$. While the partial derivatives in (25) are cumbersome to calculate, one can verify that

$$\begin{pmatrix} \theta'_{f,t} \\ \theta'_{p,t} \end{pmatrix} = \phi \mathbf{A}(\theta^*(\phi)) \begin{pmatrix} \theta'_{f,t-1} \\ \theta'_{p,t-1} \end{pmatrix}, \tag{26}$$

where we emphasize the dependence of θ^* on ϕ and write $\theta^* = \theta^*(\phi)$. With this notation, we have $\theta^*(\phi) \to \theta^*(0)$ as $\phi \to 0$ and $\theta^*(0)$ is the solution of (21) when $\phi = 0$, which coincides with (θ_f, θ_p) calculated from (5) and (6).

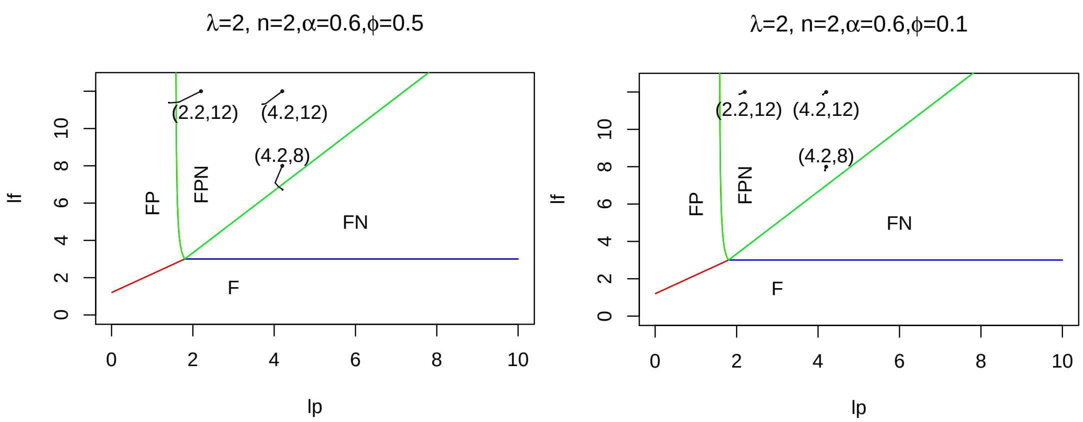
Assuming the 2×2 matrix $\mathbf{A}(\theta^*(\phi))$ to be diagonalizable, there then exists a nonsingular 2×2 matrix $\mathbf{P}(\phi)$ such that

$$\mathbf{P}(\phi)^{-1} \mathbf{A}(\theta^*(\phi)) \mathbf{P}(\phi) = \mathbf{\Lambda}(\phi), \tag{27}$$

where matrix $\mathbf{\Lambda}(\phi)$ is a diagonal matrix and displays the eigenvalues of $\mathbf{A}(\theta^*(\phi))$ on its diagonal. By multiplying both sides of (27) by the scalar ϕ, we see that up to $o(\phi)$ the eigenvalues of the matrix in the linear dynamics in (26) are ϕ times the eigenvalues of $\mathbf{A}(\theta^*(\phi))$. We summarize the above analysis in the following proposition:

Proposition 9 (Local Stability of $\mathbb{FPN}$ Equilibrium) *Up to the first order in ϕ, the eigenvalues of the matrix in (26) are ϕ times the matrix $\mathbf{A}(\theta^*(\phi))$. In* *other words, system (21) is stable to the first order if ϕ is small enough.*

Importantly, Proposition 9 implies that once we find a steady state solution of (26) when $\phi = 0$, $(\theta_f, \theta_p) \in (0,1)^2$ with $0 < \theta_f < \theta_p < 1$, i.e., if we start from an $\mathbb{FPN}$ Equilibrium, then the linear dynamic system (26) will be stable as long as ϕ is sufficiently small. Therefore, if the language learning dynamics is initiated in the $\mathbb{FPN}$-equilibrium zone, the steady state of the dynamics will stay in the $\mathbb{FPN}$-equilibrium zone as long as ϕ is small enough. At first sight, it appears that the dependence on ϕ of the matrix $\mathbf{A}(\theta^*(\phi))$ might falsify Proposition 9. Notice, however, that under modest regularity conditions, we also have $\mathbf{A}(\theta^*(\phi)) \to \mathbf{A}(\theta^*(0))$ as $\phi \to 0$, where $\mathbf{A}(\theta^*(0))$ solves (26) with $\ell_{f,t}$ and $\ell_{p,t}$ being replaced by constants ℓ_f and ℓ_p respectively.



(a) $\phi = 0.5$ (b) $\phi = 0.1$

Figure 4: Learning Dynamics (21) for $\mathbb{FPN}$-Equilibrium Zone.

We use Figure 4 to illustrate Proposition 9. Figure 4, which is based on the equilibrium map in the (ℓ_p, ℓ_f)-space in Figure 2(b), shows the trajectories of the learning dynamics of (21) from three initial points in the $\mathbb{FPN}$-equilibrium zone. For the case of $\phi = 0.5$ (Figure 4(a)), if the learning dynamics starts in the "deep" interior of the $\mathbb{FPN}$-equilibrium zone (i.e., from point $(4.2, 12)$), the steady state and the entire trajectory of the dynamics remain in the $\mathbb{FPN}$-equilibrium zone; while if the learning dynamics starts near the boundary of the $\mathbb{FPN}$-equilibrium zone (i.e., from points $(2.2, 12)$ and $(4.2, 8)$), the steady state wanders out of the $\mathbb{FPN}$-equilibrium zone. However, all the three trajectories stay entirely inside the $\mathbb{FPN}$-equilibrium zone when $\phi = 0.1$ (Figure 4(b)), which is consistent with Proposition 9.

In summary, this section focuses on the dynamics of language learning and dynamic stability of language equilibria. In the binary setting of the full and no learning, we show that a small learning speed ϕ leads to a unique and globally stable steady state, which also implies a local stability result for an interior $\mathbb{FN}$ equilibrium. Similar results hold for the $\mathbb{FP}$ equilibrium zone. For the most challenging and realistic $\mathbb{FPN}$ equilibrium, where all minority groups have three levels of language acquisition, we again demonstrate local stability under small learning speeds ϕ. Notably, a sufficiently low ϕ guarantees stability, while a high ϕ can lead to a shift from undesirable to desirable steady states (see Figure 4(a)). This insight suggests potential strategies for breaking out of suboptimal language acquisition situations and achieving qualitatively different outcomes in some language economies.

5 Asymmetric Minority Groups

We now undertake an equilibrium analysis of a language economy where multiple minority groups are asymmetric in their sizes. Our analysis is conducted in two familiar language acquisition settings: binary and ternary. In particular, we show that in a language economy with asymmetric minority groups, a larger minority group, due to the existence of larger within-group communicative benefits, acquires the majority language at a **lower** rate, or less intensively.

To begin, consider a language economy with a majority group S_0 and K minority groups, $S_1, ..., S_K$. The population size of each group is given by π_k, $k = 0, 1, ..., K$, where without loss of generality, we assume that $\pi_0 > \pi_1 > \cdots > \pi_K$. Hence, S_1 is the largest minority group, S_K is the smallest, and the total population size of the economy is $\sum_{k=0}^{K} \pi_k$. As before, only minority agents make acquisition decisions toward the majority language, with the acquisition cost parameters ℓ_f, ℓ_p and the communicative benefit of α for partial language acquisition. In addition, we assume that the language aptitude of each minority group θ is uniformly distributed, $\theta \sim \mathcal{U}[0, 1]$.

First, consider the setting where only binary language choices are available and denote the resulting game as G^B. The equilibrium group strategy, σ_k^*, for minority group S_k is defined by a group-specific cutoff $\theta_f^k \in [0,1]$. For type θ in S_k, her payoffs from $\{F, N\}$ is

$$\begin{aligned} u_k\left(F, \sigma_{-k}^*; \theta\right) &= \pi_0 + \pi_k + \sum\nolimits_{k' \geq 1, k' \neq k} \pi_{k'} \theta_f^{k'} - \ell_f \theta, \\ u_k\left(N, \sigma_{-k}^*; \theta\right) &= \pi_k, \end{aligned}$$

where the indifferent type θ_f^k is found as

$$u_k\left(F, \sigma_{-k}^*; \theta_f^k\right) = u_k\left(N, \sigma_{-k}^*; \theta_f^k\right).$$

The equilibrium cutoffs $\widehat{\theta} = \left(\theta_f^1, \ldots, \theta_f^K\right)'$ are hence determined by the following system of linear equations:

$$\begin{bmatrix} \ell_f & -\pi_2 & \cdots & -\pi_K \\ -\pi_1 & \ell_f & \cdots & -\pi_K \\ \vdots & \vdots & \ddots & \vdots \\ -\pi_1 & -\pi_2 & \cdots & \ell_f \end{bmatrix} \begin{bmatrix} \theta_f^1 \\ \theta_f^2 \\ \vdots \\ \theta_f^K \end{bmatrix} = \mathbf{A}\widehat{\theta} = \begin{bmatrix} \pi_0 \\ \pi_0 \\ \vdots \\ \pi_0 \end{bmatrix} = \pi_0$$

Since $\pi_1 > \cdots > \pi_K$, one can verify, using the row echelon form of $\mathbf{A}$, that the $K \times K$ matrix $\mathbf{A}$ has full rank and is invertible. There is hence a **unique** equilibrium with equilibrium cutoffs $\widehat{\theta}$ (implicitly) solved by

$$\widehat{\theta} = \mathbf{A}^{-1}\pi_0 \tag{28}$$

with the restriction that $\theta_f^k \in [0,1]$ for all $k \geq 1$. Example 1 provides a numerical illustration of the unique equilibrium:

Example 1 *Consider a setting where* $\pi_0 = 1$, $\pi_1 = 0.75$, $\pi_2 = 0.5$, $\pi_3 = 0.25$, $\pi_4 = 0.125$, *and the learning cost is* $\ell_f = 3$. *We can calculate that*

$$\mathbf{A} = \begin{bmatrix} 3 & -0.5 & -0.25 & -0.125 \\ -0.75 & 3 & -0.25 & -0.125 \\ -0.75 & -0.5 & 3 & -0.125 \\ -0.75 & -0.5 & -0.25 & 3 \end{bmatrix}$$

and $\widehat{\theta} = \mathbf{A}^{-1}\pi_0 = (0.49, 0.53, 0.57, 0.59)$. *Hence in equilibrium, there is less language learning in a larger minority group.*

Next consider the game G^T, where all the three language choices $\{F, P, N\}$ are available. The equilibrium group strategy, σ_k^*, for minority group S_k is then defined by two group-specific cutoff θ_f^k and $\theta_p^k \in [0,1]$ with $\theta_p^k \geq \theta_f^k$. The payoffs of type θ in S_k from $\{F, P, N\}$ can be written as:

$$\begin{aligned}
u_k\left(F, \sigma_{-k}^*; \theta\right) &= \pi_0 + \pi_k + \sum\nolimits_{k' \geq 1, k' \neq k} \pi_{k'} \left[\theta_f^{k'} + \alpha\left(\theta_p^{k'} - \theta_f^{k'}\right)\right] - \ell_f \theta, \\
u_k\left(P, \sigma_{-k}^*; \theta\right) &= \alpha\pi_0 + \pi_k + \sum\nolimits_{k' \geq 1, k' \neq k} \pi_{k'} \left[\alpha\theta_f^{k'} + \alpha^2\left(\theta_p^{k'} - \theta_f^{k'}\right)\right] - \ell_p \theta, \\
u_k\left(N, \sigma_{-k}^*; \theta\right) &= \pi_k
\end{aligned}$$

The cutoffs θ_f^k and θ_p^k can be determined from:

$$\begin{aligned}
u_k\left(F, \sigma_{-k}^*; \theta_f^k\right) &= u_k\left(P, \sigma_{-k}^*; \theta_f^k\right) \\
u_k\left(P, \sigma_{-k}^*; \theta_p^k\right) &= u_k\left(N, \sigma_{-k}^*; \theta_p^k\right)
\end{aligned}$$

The equilibrium cutoffs $\theta = \left(\theta_f^1, \ldots, \theta_f^K; \theta_p^1, \ldots, \theta_p^K\right)'$ are again determined by a system of linear equations:

$$\mathbb{A}\theta = \mathbf{\Pi}_0 \tag{29}$$

where $\mathbb{A}$ is a $2K \times 2K$ matrix defined as ($\Gamma = -(1-\alpha)^2$ and $\Phi = -(\alpha - \alpha^2)$):

$$\left[\begin{array}{cccc|cccc}
\ell_f - \ell_p & \Gamma\pi_2 & \cdots & \Gamma\pi_K & 0 & \Phi\pi_2 & \cdots & \Phi\pi_K \\
\Gamma\pi_1 & \ell_f - \ell_p & \cdots & \Gamma\pi_K & \Phi\pi_1 & 0 & \cdots & \Phi\pi_K \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\Gamma\pi_1 & \Gamma\pi_2 & \cdots & \ell_f - \ell_p & \Phi\pi_1 & \Phi\pi_2 & \cdots & 0 \\
\hline
0 & \Phi\pi_2 & \cdots & \Phi\pi_K & \ell_p & -\alpha^2\pi_2 & \cdots & -\alpha^2\pi_K \\
\Phi\pi_1 & 0 & \cdots & \Phi\pi_K & -\alpha^2\pi_1 & \ell_p & \cdots & -\alpha^2\pi_K \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\Phi\pi_1 & \Phi\pi_2 & \cdots & 0 & -\alpha^2\pi_1 & \cdots & \cdots & \ell_p
\end{array}\right] = \left[\begin{array}{c|c} \text{A} & \text{B} \\ \hline \text{C} & \text{D} \end{array}\right]$$

and $\mathbf{\Pi}_0 = ((1-\alpha)\pi_0, \ldots, (1-\alpha)\pi_0; \alpha\pi_0, \ldots, \alpha\pi_0)' = (\mathbf{\Pi}_0^1, \mathbf{\Pi}_0^2)'$.

We again use a numerical example to illustrate the game G^T:

Example 2 *Consider a setting where* $\pi_0 = 1$, $\pi_1 = \frac{1}{2}$, $\pi_2 = \frac{1}{4}$, $\alpha = 0.5$ *and* $\ell_f = 4$, $\ell_p = 1$. *It can be verified that the linear equation system is*

$$\begin{bmatrix} 3 & -\frac{1}{16} & 0 & -\frac{1}{16} \\ -\frac{1}{8} & 3 & -\frac{1}{8} & 0 \\ 0 & -\frac{1}{16} & 1 & -\frac{1}{16} \\ -\frac{1}{8} & 0 & -\frac{1}{8} & 1 \end{bmatrix} \begin{bmatrix} \theta_f^1 \\ \theta_f^2 \\ \theta_p^1 \\ \theta_p^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} \theta_f^1 \\ \theta_f^2 \\ \theta_p^1 \\ \theta_p^2 \end{bmatrix} = \begin{bmatrix} \frac{13}{71} \\ \frac{14}{71} \\ \frac{39}{71} \\ \frac{42}{71} \end{bmatrix}.$$

Hence, both minority groups choose full learning, partial learning, and no learning in equilibrium, with the smaller minority group (S_2) *acquiring the dominant language more intensively.*

We now state our main result for the setting of asymmetric minority groups:

Proposition 10 (More Acquisition for Smaller Minority Groups) *Consider the language economy with* K *minority groups, with population sizes* $\pi_0 > \pi_1 > \cdots > \pi_K$, *and language equilibria determined by* (28) *and* (29).

1. *In game* G^B, *if* ℓ_f *is sufficiently large, then we have that in equilibrium*

$$0 < \theta_f^1 < \theta_f^2 < \cdots < \theta_f^K < 1.$$

2. *In game* G^T, *if* α, ℓ_f *and* ℓ_p *are sufficiently large with* $\ell_f > \ell_p > 0$, $\alpha \in (0, 1)$, *then we have that in equilibrium,* $0 < \theta_f^i < \theta_p^i < 1\ \forall i$ *and*

$$\theta_f^1 < \theta_f^2 < \cdots < \theta_f^K \text{ and } \theta_p^1 < \theta_p^2 < \cdots < \theta_p^K.$$

Remark 2 *In Proposition 10, we have only examined* ***interior*** *equilibria in the games* G^B *and* G^T. *In particular, we have focused on leading terms while neglecting insignificant ones in our reasoning to obtain a qualitative result that, in general, smaller minority groups acquire the majority language more intensively in equilibrium. More precise properties of the language equilibrium in the asymmetric setting are difficult, if not impossible, to obtain.*

6 Econometrics

In this section, we consider the identification of the social effects that determine equilibrium language acquisition. Our model does not have a direct statistical generalization, so our objective here is to characterize how one can obtain evidence for the mechanisms that underlie our model.

Assume that agents are randomly drawn from a set of neighborhoods. We denote an agent as i and her neighborhood as $n(i)$. Here, "neighborhoods" could be census blocks, census tracts, or even larger population units. Suppose that data is available not only on individual agent neighborhood locations and associated choices F vs P vs N, but also on observable covariates describing agent i and observable covariates describing various aspects of $n(i)$, along with measures of language choices within the neighborhoods.

Our econometric model treats ability to learn the majority language and levels of language fluency in the majority language as functions of observable covariates. It is natural to work with a measure of skill in our econometric model, so we replace the (discretized) cutoffs in θ space, $0 < \theta_{\min} < \theta_2 < \cdots < \theta_I < 1$, with cutoffs in a language learning skill measure $S = 1/\theta - 1$, with $\infty > S_{\max} > \cdots > S_1 > 0$. We maintain Assumption 2 for the equilibrium $\mathbb{FPN}$, with analogous assumptions for the equilibria $\mathbb{F}, \mathbb{FN}$ and $\mathbb{FP}$:

Assumption 2 (Fixing Cutoffs) *Set* $S_{f,n(i)} = \frac{1}{\theta_{f,n(i)}} - 1$ *and* $S_{p,n(i)} = \frac{1}{\theta_{p,n(i)}} - 1$ *where* $\theta_{f,n(i)}$, $\theta_{p,n(i)}$ *solve (5) and (6) in Section 3.2 as the equilibrium cutoffs for neighborhood* $n(i)$ *with* $\lambda = \lambda_{n(i)}$.

$S_{f,n(i)}$ and $S_{p,n(i)}$ are the respective learning-skill cutoffs for full learning and partial learning in neighborhood $n(i)$. We assume that these cutoffs are observable to the econometrician and our data set is rich enough to include as many neighborhoods as needed to get enough variation for our identification analysis below. Indeed, if the data set at the census tract or census block level is rich enough to have measures of the fractions of non-learners, partial learners, and full learners, it is then possible to approximate the cutoffs from the data. To be explicit, one can construct the cutoffs using the observable fractions of full learners, partial learners, and non-learners in neighborhood $n(i)$, denoted respectively as $Z_{F,n(i)}$, $Z_{P,n(i)}$, and $Z_{N,n(i)}$, i.e.,

$$\begin{aligned} Z_{F,n(i)} &= \mu\left\{i \in n(i) \,|S_i \geq S_{f,n(i)}\right\} \equiv \left[F_{S,n(i)}\left(S_{\max,n(i)}\right) - F_{S,n(i)}\left(S_{f,n(i)}\right)\right], \\ Z_{P,n(i)} &= \mu\left\{i \in n(i) \,|S_{f,n(i)} > S_i \geq S_{p,n(i)}\right\} \equiv \left[F_{S,n(i)}\left(S_{f,n(i)}\right) - F_{S,n(i)}\left(S_{p,n(i)}\right)\right], \\ Z_{N,n(i)} &= \mu\left\{i \in n(i) \,|S_{p,n(i)} > S_i\right\} = 1 - Z_{F,n(i)} - Z_{P,n(i)}, \end{aligned}$$

where $\mu\{A\}$ denotes the measure of the set A and $F_{S,n(i)}$ is the corresponding cumulative empirical distribution of learning skills from measure $\mu\{\cdot\}$.

We consider the econometric model:

$$S_i = k + c'\mathbf{X}_i + d'\mathbf{Y}_{n(i)} + J_F Z_{F,n(i)} + J_P Z_{P,n(i)} + J_N Z_{N,n(i)} + \eta_i. \quad (30)$$

The terms $\mathbf{X}_i, \mathbf{Y}_{n(i)}$, and η_i are, respectively, an r-dimensional vector of observed individual covariates ($\mathbf{X}_i$), an s-dimensional vector of observed "contextual" covariates for neighborhood $n(i)$ ($\mathbf{Y}_{n(i)}$), and regression errors (η_i), while $Z_{F,n(i)}$, $Z_{P,n(i)}$, and $Z_{N,n(i)}$ are the observed fractions of F-, P-, and N-learners defined above.

Throughout, we assume the unobserved heterogeneity in the system is orthogonal to the observable determinants of skill:

Assumption 3 (Orthogonality of η_i) $\mathbb{E}\left\{\eta_i|\mathbf{X}_i, \mathbf{Y}_{n(i)}\right\} = 0$.

This assumption allows us to focus on the specific identifications of social models such as (30); we discuss relaxation of this assumption below.

Equation (30) is a variation of the standard model of social interactions (see Manski (1993)[48] for the original formulation and Section 3.2 of Brock and Durlauf (2001b)[10] for the general version). Relative to the original Manski model, this formulation allows neighborhood variables to differ from averages of the individual-level variables and allows for nonlinearities in feedback as in equation (30) by incorporating the fractions of $Z_{F,n(i)}$, $Z_{P,n(i)}$, and $Z_{N,n(i)}$ as additional regressors, with the restriction that the sum of the three fractions adds up to one in each neighborhood $n(i)$.

The objective of this section is to ask whether parameters mapping $Z_{F,n(i)}$, $Z_{P,n(i)}$, and $Z_{N,n(i)}$ to language proficiency are identified. Identification issues are raised by *the reflection problem* (Manski 1993[48]), which is a variant of the identification problem in rational expectations econometrics, e.g., Wallis (1980)[57], in that it involves potential collinearity between expected values which drive behavior and other variable present in the equation. To understand when identification holds or fails, we follow the same procedure as in Brock and Durlauf (2001a,b)[9][10].

We start our identification analysis with the simplest binary language acquisition case: F vs N with P not possible. We assume that $S_i \leq S_{\max} < \infty$, $\theta_i \geq \theta_{\min} > 0$. Following (30), for this case, we have:

$$\begin{aligned} S_i &= k + c'\mathbf{X}_i + d'\mathbf{Y}_{n(i)} + J_F Z_{F,n(i)} + J_N \left(1 - Z_{F,n(i)}\right) + \eta_i \\ &= k + J_N + c'\mathbf{X}_i + d'\mathbf{Y}_{n(i)} + \left(J_F - J_N\right) Z_{F,n(i)} + \eta_i. \end{aligned} \tag{31}$$

Following Brock and Durlauf (2001b)[10], suppose that the linear space spanned by $\left(1, \mathbf{X}_i, \mathbf{Y}_{n(i)}, Z_{F,n(i)}\right)$ is $r+s+2$, where recall that $\mathbf{X}_i$ has dimension r and $\mathbf{Y}_{n(i)}$ has dimension s. There are two composite constants (i.e., $k + J_N$ and $J_F - J_N$) and two vectors of dimensions r and s (i.e., c' and d') for a total of $r+s+2$ objects to identify. Hence, we know that $(k + J_N)$, c', d', $(J_F - J_N)$ can be identified. A problem however remains in that we have three constants k, J_F, J_N but only two equations (i.e., the identified "$k+J_N$" and "$J_F - J_N$") to solve for k, J_F and J_N. As a result, one of the constants in (k, J_N, J_F) remains *unidentified*. This limit does not mean that the data are informative as whether (J_N, J_F) are both zero. Second, knowledge about the magnitude of language spillover effects has natural policy value due to social multipliers they produce with respect to policy interventions to raise language skill levels.

For the general case of F vs P vs N, i.e., with partial language acquisition, an analogous argument holds. Recall that $Z_{F,n(i)}$, $Z_{P,n(i)}$, (30), and equations (5), (6) of Section 3.2 above for the formulas for the cutoffs:

$$S_i = k + J_N + c'\mathbf{X}_i + d'\mathbf{Y}_{n(i)} + \left(J_F - J_N\right) Z_{F,n(i)} + \left(J_P - J_N\right) Z_{P,n(i)} + \eta_i. \tag{32}$$

Theorem 1 below presents our identification results for equation (32):

Theorem 1 *Assume that the dimension of the linear space spanned by the elements of*

$$\left(1, \mathbf{X}_i, \mathbf{Y}_{n(i)}, Z_{F,n(i)}, Z_{P,n(i)}\right)$$

is at least $3 + r + s$*, then the parameters* $k + J_N$*,* c'*,* d'*,* $J_F - J_N$*,* $J_P - J_N$ *are identified.*

Theorem 1, a natural consequence of Theorem 6 of Brock and Durlauf (2001b)[10], shows that the previous positive identification results apply to our three choice framework. Theorem 1 has an immediate corollary:

Corollary 1 *The parameters* c' *and* d' *are identified. The composite parameters* $k + J_N$*,* $J_F - J_N$*,* $J_P - J_N$ *are identified.*

To understand Corollary 1, notice that c' is identified by the dimension r of the linear space spanned by $\mathbf{X}_i$, while d' is identified by the dimension s of the linear space spanned by $\mathbf{Y}_{n(i)}$. In addition, since the three composite parameters are used to pin down four constants (k, J_F, J_P, J_N), one of them remains unknown. As before, the identification is partial.

The non-identification of the constants $(k,\ J_F,\ J_P,\ J_N)$ in our setting is, in our view, not a serious drawback. After all, the composite parameters $(J_F - J_N)$ and $(J_P - J_N)$, i.e., the partial derivatives of skill S_i with respect to $Z_{F,n(i)}$ and $Z_{P,n(i)}$, are identified. Intuitively, these composite parameters measure the externalities of the "aggregate" language acquisition behavior in neighborhood $n(i)$ on an individual's language skill and hence her language acquisition behavior. Knowledge about the magnitude of such externalities indeed offers useful information for policy makers, and hence is, in our view, of primary policy interest.

What drives the identification result? The key substantive requirement is that $Z_{F,n(i)}$ is linearly independent of $\left(1, \mathbf{X}_i, \mathbf{Y}_{n(i)}\right)$. There are many routes to such linear independence. For example, linear independence of $Z_{F,n(i)}$ over $\left(1, \mathbf{X}_i, \mathbf{Y}_{n(i)}\right)$ can be achieved if $\lambda_{n(i)}$, the relative population size of the majority in $n\,(i)$, varies independently of $\left(1, \mathbf{X}_i, \mathbf{Y}_{n(i)}\right)$, which is indeed plausible. More generally, $Z_{F,n(i)}$ is generically a nonlinear function of the joint density of

$\left(1, \mathbf{X}_i, \mathbf{Y}_{n(i)}\right)$ in the sense that the set of densities η_i that produce linear dependence is nongeneric in the space of densities that are absolutely continuous. See Brock and Durlauf (2007)[13] for discussion of this point.

A major limitation to the above findings is Assumption 3, for the obvious reason that it ignores endogeneity of neighborhood membership. However, there is a constructive route to identification if one models self-selection via the construction of control function variables, cf. Heckman (1979)[39], to augment identification. Note that semiparametric estimates will suffice for identification. To see this, suppose agent i is observed in neighborhood $n(i)$ if and only if a latent variable $t_i > 0$ exists, where t_i measures i's evaluation of $n(i)$, and can be written as a linear function of a vector of observables $(\mathbf{R}_i)$ and a normally distributed error τ_i, i.e., $t_i = \gamma' \mathbf{R}_i + \tau_i$. Assume that the error τ_i and the regression error η_i in equations (31) and (32) are jointly normally distributed. Then following Section 3.6 of Brock and Durlauf (2001b)[10], we obtain two new regressors at the price of one extra parameter. This approach can be useful if there is enough variation in the average over $n(i)$ of the control function variable across the neighborhoods in the data. The analyst also needs to find a regressor to include in $\mathbf{R}_i$ that is not already in the primary regression before correction for selection bias.[37] But this requirement is standard when addressing self selection. The upshot of our discussion is that self-selection of neighborhoods does not raise any new issues in the context of our language model.

7 Conclusions and Future Research

This paper presented a theoretical language acquisition framework where individuals from multiple minority groups can choose to learn the majority language at three different levels of fluency: fluent, partially fluent, and not fluent at all. An important feature of our framework is the existence of positive externalities

[37] This approach to incorporating selection bias approach in the identification of social interactions has been implemented in previous studies, perhaps first by Ioannides and Zabel (2008)[42]. See also Sheng and Sun (2022)[54] for recent work along this line that extends Brock and Durlauf (2001b, Section 3.6)[10] to two-sided matching models of group formations, suggesting richer models of social structure and language acquisition may be studied.

for the whole economy in language learning. We showed that such externalities can generate multiple language equilibria in a general setting.

Our theory development on language acquisition was followed by a dynamic analysis, with a main purpose of investigating local stability of the equilibria found in the static language acquisition framework. In particular, we considered a deterministic learning dynamic process where the costs of language learning adjust over time in accordance with how many minority agents partially or fully learn the majority language in the previous period. We found that depending on the adjustment rate of the learning costs, there could be locally stable or locally non-stable equilibria. Our analysis here helps us understand what structural features are important for stability, as well as limiting configurations of language acquisition behavior in our framework.

Finally, we showed how our model can be related to empirical work by exploring how language spillovers of the type we study may be uncovered empirically. Here we argue that our conceptual framework leads to positive identification results under empirically plausible conditions.

In terms of future research, we see value in integrating neighborhood choice and language choice into a common framework. One route to this would be via a sequential logit approach in which individuals first choose neighborhoods and then choose F vs P vs N in an empirical following ordered logit framework. Recall that we have two thresholds $S_{p,n(i)} < S_{f,n(i)}$ where agents in $n(i)$ choose N for $S_i \leq S_{p,n(i)}$, choose P for $S_{p,n(i)} < S_i \leq S_{f,n(i)}$, and choose F for $S_{f,n(i)} < S_i$. This integration can lead to more complicated dynamics when one considers the coevolution of neighborhood memberships and language choice.

A second research direction involves using our framework to systematically investigate the sources of heterogeneity in partial language versus full language acquisition. For example, our model would explain the Belgium and US steady-state differences by focusing on limited communicative benefits available to those acquire Flemish and French in Belgium as compared to the extensive communication and market reach to learners of English in the US. Moreover, partial learning, linguistic interaction between English and Spanish, and the emergence of Spanglish in the US, are different from the relatively static co-existence of

Flemish and French, highlighting a different linguistic dynamics. Evaluating whether these differences in fact produce the language patterns we discuss requires moving toward structural empirical work.

Finally, recall that language equilibria typically exhibit inefficient learning compared to the socially optimal level of language acquisition. The suboptimality of equilibrium levels of language acquisition and persistence of partial learning in various censuses call for a careful and systematic analysis of public policies in this regard, which have already been discussed previously. While language can be seen as purely a communicative protocol and language acquisition can be regarded as a human capital investment so that identity considerations are not a necessary ingredient for useful economic insights, in various contexts, language acquisition and usage are heavily impacted by individuals' social identity, as "language cannot be legislated; it is the freest, most democratic form of expression of the human spirit" (Stavans (2000)[55], p.557). Thus, it would be important to address linkages between language learning and identity in various settings. An important next step in developing these models is the introduction of identity considerations in the spirit of Bisin and Verdier (2000)[14] as well as in the spirit of Laitin (1993)[44]. To do this requires a distinct formulation of the utility of identity, the meaning of solidarity of co-ethnics as such, and should not amount to more than simply adding percentages of co-ethnic learners in the utility function. Marrone (2019)[50] gives a variation of this type of approach in considering identity and language investment as joint processes. Our proposal is to treat economic benefits and identity benefits as distinct processes. For this reason, we pursue that approach in a sequel paper.

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Everybody's Talkin' at Me: Levels of Majority Language Acquisition by Minority Language Speakers

Online Appendix

William A. Brock, Bo Chen, Steven N. Durlauf, Shlomo Weber

[For Online Publication]

This Appendix contains two sections. Section 8 provides additional census information for Table 1 in the paper (subsection 8.1), as well as explicit language requirements for permanent residency and citizenship from the "Report on the 2018 Council of Europe and ALTE survey on language and knowledge of society policies for migrants" (subsection 8.2). Section 9 contains proofs and additional illustrations that were omitted from the paper due to space constraints.

8 Partial Learning in Various Countries

8.1 Partial Learning in Censuses

In order to offer an empirical support for our claim that partial learners represent a sizeable percent among those who do not speak a majority language, we now present a brief examination of the degree of command of English for the group of partial learners across the United States, the United Kingdom, Ireland, and Australia.

To create statistics about language and the ability to speak English, all US censuses since 1890 (with exception of the 1950 census) contained questions about whether a person speaks a language other than English at home, what language he/she speaks, and how well he/she speaks English. While in earlier censuses the ability to speak English was coded as yes or no, since the 1980 census, however, the command of English for those who do not speak English at home was categorized by four possible options: (i) speaking it very well (group E), (ii) speaking well (Group F), (iii) speaking not well (group G), (iv) not speaking at all (group C). That is, the recognition of an incomplete or partial command of English has become prominent already 40 years ago. Groups F and G jointly contain about 34% of those who do not speak English at home. If we identify partial learners as members of group G only (those who speak English not well), the number is still substantial–about 14%.[38] By using this census

[38] S. Ruggles, S. Flood, S. Foster, R. Goeken, J. Pacas, M. Schouweiler and M. Sobek (2021). IPUMS USA: Version 11.0 [dataset]. Minneapolis, MN: IPUMS. https://doi.org/10.18128/D010.V11.0.

data, Carliner (2000)[15] points out quite different earning patterns of these groups. For example, among well-educated men, those who speak English very well earn 9.6% more than men who speak English well, 17.6% more than men who speak English poorly, and 33.6% more than men who speak no English.

By applying the same methodology to the 2016 census in Ireland, the same two groups E and F yield the 45% from the total number of residents of Ireland who do not speak English at home, while the group G alone represents about 13% of those respondents.[39]

The data for UK does not distinguish between those who speak English well and very well. In the US census terminology, the fraction of those who do not speak English well reaches 17.5%.[40]

Similarly to the UK data, the Australian census lumps together those who speak English well and very well. Moreover, it does not distinguish between those who speak English not well or not at all. The fraction of the latter group among all those who do not speak English at home turns out to be 20%.[41]

8.2 Explicit or Implicit Ban on Partial Learning

In numerous countries of the world, ban on partial learning exhibits itself via obligatory tests in host country language for immigrants when applying for citizenship. Although such tests don't necessarily require a truly high level of proficiency in the language, they still act as a ban or restriction on partial learning. An important question is what should be the cutoff levels to distinguish between partial and full degrees of command? It is obviously a judgement call, but it seems that according to the official Common European Framework of References (CEFR) classification of language skills (from A1 to C2), a country requiring at least B1 (lower-intermediate) can already be implicitly considered as a country with a ban on partial language learning.[42] The Council of Europe has conducted another round of survey among European countries' migration officials in 2018.[43] There are various language requirements for entry, temporary and permanent residency in various countries. It is worth pointing out that many countries impose quite stringent linguistic barriers for residency and

[39] Ireland, Census 2016. Speakers of Foreign Languages by Ability to Speak English, NUTS 3, Theme 2.6, Ireland, 2016, CSO & OSi. Link: Ireland 2016 Census Accessed: 5/4/22.

[40] England and Wales, Census 2011. Proficiency in English by year of arrival in the UK by country of birth (national). Nomis, UK, 2011. Link: UK 2011 Census Accessed: 5/4/22.

[41] Data Informed Decisions. Australia, Proficiency in English [table], Australia. `https://profile.id.com.au/australia/speaks-english` Accessed: 5/4/22. The data is compiled from Australian Bureau of Statistics, Census of Population and Housing 2011 and 2016.

[42] For official descriptions of CEFR levels, see Common Reference Levels for Languages.

[43] For a detailed report about the survey, see Linguistic Integration of Adult Migrants.

citizenship, which become more prohibitive over time. To illustrate the point, Table 3 presents the language requirements for permanent residency in various countries included in the Council of Europe survey in 2018:

Country	Listening	Reading	Speaking	Writing
Austria	A2	A2	A2	A2
Belgium (Fl.)	A2	A2	A2	A2
Cyprus	A2	A2	A2	A2
Czech Republic	A1	A1	A1	A1
Denmark	B1	B1	B1	B1
France	A2	A2	A2	A2
Germany	B1		B1	
Greece	A2	A2	A2	A2
Iceland	Unspecified			
Italy	A2	A2	A2	A2
Lithuania	Unspecified			
Luxembourg	A2	A2	A2	A2
Malta	Unspecified			
Netherlands	A2	A2	A2	A2
North Macedonia	Unspecified			
Norway			A1	
Portugal	A2	A2	A2	A2
Russia	A2	A2	A2	A2
Switzerland	A2	A1	A2	A1
UK	B1	B1	B1	B1

Table 3. Language Requirements for Permanent Residency in Europe (2018).

And Table 4 shows the language requirements for citizenship in various countries in the same survey in 2018:

Country	Listening	Reading	Speaking	Writing
Albania	Unspecified			
Armenia	Unspecified			
Austria	B2	B2	B2	B2
Belgium (Fl.)	A2	A2	A2	A2
Belgium (Fr.)	A2	A2	A2	A2
Croatia	Unspecified			
Czech Republic	B1	B1	B1	B1
Denmark	B2	B2	B2	B2
Finland	B1	B1	B1	B1
France	B1	B1	B1	B1
Germany	B1	B1	B1	B1
Greece	B2	B2	B2	B2
Hungary	Unspecified			
Iceland	B1	B1	B1	B1
Italy	B1	B1	B1	B1
Latvia	Unspecified			
Lithuania	Unspecified			
Luxembourg	B1		A2	
Malta	Unspecified			
Moldova	B2	B2	B2	B2
Netherlands	A2	A2	A2	A2
North Macedonia	Unspecified			
Norway			A2	
Poland	B1	B1	B1	B1
Portugal	A2	A2	A2	A2
Romania	A1	A1	A1	A1
Russian Federation	A2	A2	A2	A2
Slovak Republic	Unspecified			
Slovenia	A2	A2	A2	A2
Spain	A2	A2	A2	A2
Switzerland	B1	A2	B1	A2
Turkey	Unspecified			
UK	B1	B1	B1	B1

Table 4. Language Requirements for Citizenship in Europe (2018).

9 Additional Illustrations and Proofs

9.1 Equilibrium Multiplicity

We construct here an explicit numerical example to show that multiple equilibria, yielding different learning outcomes, are a real, not just conceptual, phenomenon in our language economy. And we do this for both the binary acquisition setting and the partial acquisition setting, with a focus on equilibrium $\mathbb{FPN}$ for the latter.

Consider the following piecewise linear distribution:

$$\hat{H}(\theta) = \begin{cases} x\theta, & \text{for } \theta \in \left[0, \frac{1}{4}\right] \\ y\theta - \frac{y-x}{4} & \text{for } \theta \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ z\theta + \frac{3(y-z)-(y-x)}{4} & \text{for } \theta \in \left[\frac{3}{4}, 1\right] \end{cases}, \text{ where } \begin{array}{c} x, y, z > 0, y > x, \\ y > z, \text{ and } 2y + x + z = 4. \end{array}$$

Hence, $\hat{H}(\theta)$ has three connected linear segments, with a reverse Z shape.

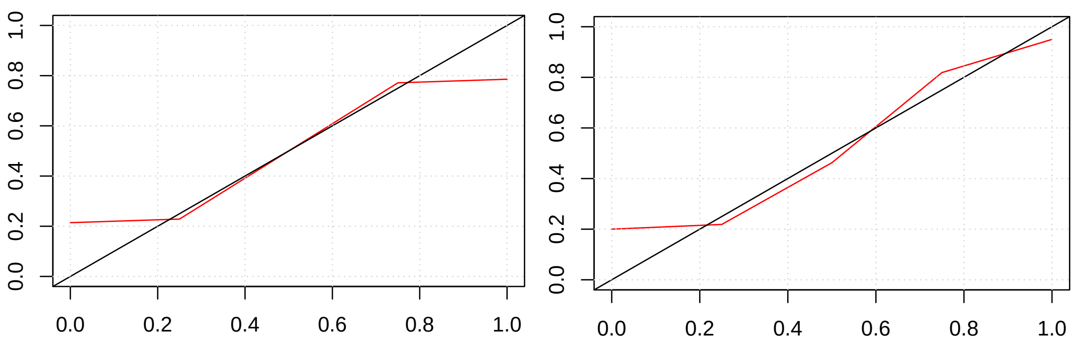


Figure 5: Equilibria under $\hat{H}(\theta)$ for Binary & Partial Language Acquisition.

Figure 5 illustrates the equilibrium characterizations for the binary acquisition setting (left panel) and the partial acquisition setting (right panel), using equilibrium conditions (2) and (5)-(6) respectively.[44]

From a technical point of view, multiple language equilibria can arise in our language economy because there is essentially no restriction on the density

[44] In each panel, the horizontal axis denotes θ, while the vertical axis denotes function $\left[\lambda + (n-1)\hat{H}(\theta_f)\right]/\ell_f$ (left) and $\left[\alpha\lambda + \alpha^2(n-1)H(\theta_p) + \alpha(1-\alpha)(n-1)H\left(\frac{\ell_p(1-\alpha)}{\alpha(\ell_f-\ell_p)}\theta_p\right)\right]/\ell_p$ (right) where we have implicitly used the relationship in (8). The parameters used in Figure 5 are $x = 0.1$, $y = 1.9$, $z = 0.1$, $\ell_f = 9$, $\ell_p = 3$, $\alpha = 0.5$, $\lambda = 1.2$ and $n = 7$.

function of $H(\theta)$. In particular, such a flexible density can generate various patterns for the slopes of (the right-hand sides of) the equilibrium conditions (2) and (5)-(6), resulting in equilibrium multiplicity.

9.2 Algebraic Representation for Figure 2(a)

Figure 2(a) is constructed with parameters: $\lambda = 2, n = 2, \ell_p = 1$. According to Proposition 3, the parameter constellations for the four equilibrium formats are algebraically as follows:

$$\begin{aligned}
\text{Equilibrium } \mathbb{F} &: \quad \ell_f \leq \min\{3, (4-3\alpha)\} \\
\text{Equilibrium } \mathbb{FP} &: \quad \left\{ \begin{array}{l} \text{If } \alpha > \sqrt{2}-1 > \frac{1}{3}, \text{ then } \ell_f > 4-3\alpha; \\ \text{If } \frac{1}{3} < \alpha \leq \sqrt{2}-1, \text{ then } 4-3\alpha < \ell_f < \frac{2(1-2\alpha)}{1-2\alpha-\alpha^2}. \end{array} \right\} \\
\text{Equilibrium } \mathbb{FN} &: \quad \alpha \leq \frac{1}{3},\ 3 < \ell_f \leq \frac{1}{\alpha}. \\
\text{Equilibrium } \mathbb{FPN} &: \quad \left\{ \begin{array}{l} \ell_f > \frac{2}{1+\alpha}, \ell_f > \frac{1}{\alpha}, \text{ and} \\ \quad \text{If } 0 < \alpha < \sqrt{2}-1, \text{ then } \ell_f > \frac{2(1-2\alpha)}{1-2\alpha-\alpha^2} \\ \quad \text{If } \alpha \in \left[\sqrt{2}-1, \frac{1}{2}\right], \text{ then no solution} \\ \quad \text{If } \alpha > \frac{1}{2}, \text{ then no solution} \end{array} \right\}
\end{aligned}$$

9.3 Language Learning Dynamics: F vs P

We now analyze the learning dynamics for the $\mathbb{FP}$-equilibrium zone, which has been omitted from the main text. As in Section 4.1, we can alternatively think of the dynamic setting here as one where minority agents can only choose from $\{F, P\}$, perhaps because a government imposes a penalty for not at least partially learning the majority language so that no minority agent chooses N. Consider an initial point that is in the interior of the $\mathbb{FP}$-equilibrium zone. We will demonstrate that the dynamics with such an initial point remain in the (interior) $\mathbb{FP}$-equilibrium zone and thus an $\mathbb{FP}$ equilibrium is locally stable, as long as ϕ is sufficiently small.

Given the period-$(t-1)$ cutoffs $\theta_{f,t-1}$, $\theta_{p,t-1}$ ($\theta_{f,t-1} < 1$ and $\theta_{p,t-1} = 1$) and the rational expectation that all minority agents adopt the equilibrium cutoffs $\theta_{f,t}$ and $\theta_{p,t}$ in period t, the period-t payoffs from F, and P for a type-θ minority agent are respectively:

$$\begin{aligned}
u^t(F, \theta_{f,t-1}; \theta) &= 1 + \lambda + (n-1)\theta_{f,t} + \alpha(n-1)(\theta_{p,t} - \theta_{f,t}) - \ell_f e^{-\phi q_{f,t-1}}\theta, \\
u^t(P, \theta_{f,t-1}; \theta) &= 1 + \alpha\lambda + \alpha(n-1)\theta_{f,t} + \alpha^2(n-1)(\theta_{p,t} - \theta_{f,t}) - \ell_p e^{-\phi q_{p,t-1}}\theta.
\end{aligned}$$

where $q_{f,t-1} = \theta_{f,t-1}$, $q_{p,t-1} = \theta_{p,t-1} - \theta_{f,t-1} = 1 - \theta_{f,t-1}$.

Since $\theta_{p,t-1} = 1$ and the binary choices $\{F, P\}$, the behavior of minority agents in period t is then captured by the cutoff type $\theta_{f,t}$ who is indifferent between F and P:

$$\theta_{f,t} = \frac{(1-\alpha)[\lambda + \alpha(n-1)]}{\ell_f e^{-\phi\theta_{f,t-1}} - \ell_p e^{-\phi\left(1-\theta_{f,t-1}\right)} - (1-\alpha)^2(n-1)} \equiv g(\theta_{f,t-1}),$$

where $g(\cdot)$ is the corresponding dynamic driver function and

$$\begin{aligned} g(0) &= \frac{(1-\alpha)[\lambda + \alpha(n-1)]}{\ell_f - \ell_p e^{-\phi} - (1-\alpha)^2(n-1)}, \\ g(1) &= \frac{(1-\alpha)[\lambda + \alpha(n-1)]}{\ell_f e^{-\phi} - \ell_p - (1-\alpha)^2(n-1)}. \end{aligned}$$

We next assume

Assumption 4 $0 < g(0) < g(1) < 1$, *or*
$\ell_f > \max\left\{\ell_p e^{-\phi} + (1-\alpha)^2(n-1), \ell_p e^{\phi} + (1-\alpha)(\lambda + n - 1)e^{\phi}\right\}$.

One can again verify that under Assumption 4, the dynamic driver function $g(\cdot)$ is positive, strictly increasing and strictly convex on $[0, 1]$.

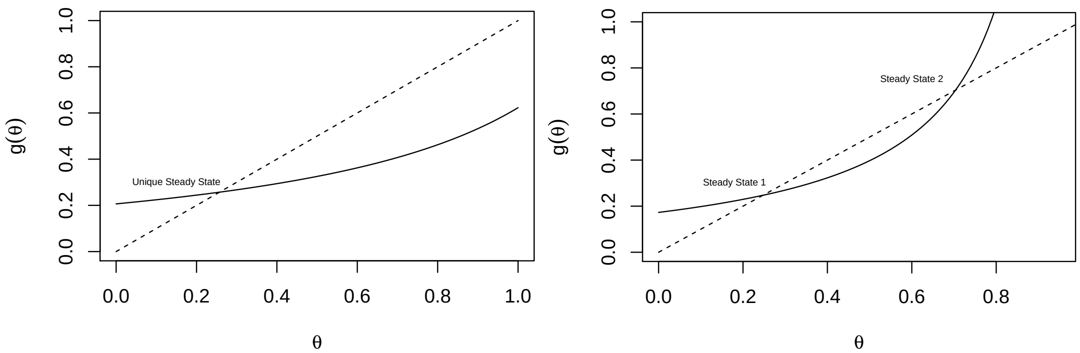
A steady state is similarly an acquisition outcome $\theta^* \in [0, 1]$ such that

$$\theta^* = g(\theta^*) = \frac{(1-\alpha)[\lambda + \alpha(n-1)]}{\ell_f e^{-\phi\theta^*} - \ell_p e^{-\phi(1-\theta^*)} - (1-\alpha)^2(n-1)}. \tag{33}$$

Assumption 4 then implies that there is a unique (interior) steady state θ^* with $\theta^* = g(\theta^*)$ and $\left.\frac{dg(\theta)}{d\theta}\right|_{\theta=\theta^*} < 1$, i.e., the unique steady state is *globally stable.* We hence have:

Proposition 11 *Suppose Assumption 4 holds. In the binary learning dynamics between F and P, there is a unique steady state $\theta^* = g(\theta^*)$ with $\theta^* \in (0, 1)$. Moreover, the unique steady state is stable.*

We similarly use the following Figure 6 to illustrate Proposition 11 where each dotted line is the 45-degree line and each solid curve is $g(\theta)$:



(a) Unique Steady State (b) Multiple Steady States

Figure 6: Binary Language Learning Dynamics: F vs P.

Figure 6(a) presents a scenario with a unique steady state as in Proposition 11 ($\lambda = 2$, $n = 2$, $\alpha = 0.6$, $\ell_f = 8$, $\ell_p = 4$, $\phi = 0.3$), while Figure 6(b) presents one with a stable steady state 1 and an unstable steady state 2 ($\lambda = 2$, $n = 2$, $\alpha = 0.6$, $\ell_f = 8$, $\ell_p = 4$, $\phi = 0.7$). One can verify that Assumption 4 is violated for Figure 6(b).

Finally, observe that if $\phi = 0$, Assumption 4 coincides with the interior $\mathbb{FP}$ equilibrium condition (see (10)) , which implies that minority agents play the (static) equilibrium $\mathbb{FP}$ in our setting with partial acquisition. Hence, if the dynamics starts inside the $\mathbb{FP}$-equilibrium zone, then Assumption 4 holds and Proposition 11 then implies that the dynamics will remain in the $\mathbb{FP}$-equilibrium zone and converge to a steady state that is close to the point where the dynamics is initiated, as long as ϕ is sufficiently small. In other words, an $\mathbb{FP}$ equilibrium in the interior of the $\mathbb{FP}$-equilibrium zone is locally stable under our dynamics for sufficiently small ϕ.

9.4 A Detailed Analysis of Dynamics in the $\mathbb{FPN}$-Equilibrium Zone

To better understand the forces behind the local stability of a steady state of the dynamic system initiated at the $\mathbb{FPN}$-equilibrium zone, i.e., the linear dynamic system (21), we consider here a special case where all n minority groups are lumped into one "ethnic" group, i.e., all minority groups are homogenous so that $n = 1$. As we will see, while it removes the interesting economics of externality, the dynamic analysis for the special setting is more transparent and intuitive.

We first rewrite the dynamic system (21) in this special setting as:

$$\begin{pmatrix} \theta_{f,t} \\ \theta_{p,t} \end{pmatrix} = \begin{pmatrix} \frac{\lambda(1-\alpha)}{\ell_f e^{-\phi\theta_{f,t-1}} - \ell_p e^{-\phi\left(\theta_{p,t-1}-\theta_{f,t-1}\right)}} \\ \frac{\alpha\lambda}{\ell_p e^{-\phi\left(\theta_{p,t-1}-\theta_{f,t-1}\right)}} \end{pmatrix} \tag{34}$$

And hence a steady state (θ_f^*, θ_p^*) of the dynamic system is represented as:

$$\begin{pmatrix} \theta_f^* \\ \theta_p^* \end{pmatrix} = \begin{pmatrix} \frac{\lambda(1-\alpha)}{\ell_f e^{-\phi\theta_f^*} - \ell_p e^{-\phi\left(\theta_p^*-\theta_f^*\right)}} \\ \frac{\alpha\lambda}{\ell_p e^{-\phi\left(\theta_p^*-\theta_f^*\right)}} \end{pmatrix}$$

To analyze local stability issues for the steady state (θ_f^*, θ_p^*), we employ a perturbation method by perturbing the initial condition $(\theta_{f,0}, \theta_{p,0})$ and differentiating the dynamic system (34) with respect to $(\theta_{f,0}, \theta_{p,0})$ to obtain the following *first variation equation*:[45]

$$\begin{aligned} \begin{pmatrix} \frac{\partial\theta_{f,t}}{\partial\theta_{f,0}} \\ \frac{\partial\theta_{p,t}}{\partial\theta_{p,0}} \end{pmatrix} &= \phi\mathbf{M}\left(\theta_{t-1}\right) \begin{pmatrix} \frac{\partial\theta_{f,t-1}}{\partial\theta_{f,0}} \\ \frac{\partial\theta_{p,t-1}}{\partial\theta_{p,0}} \end{pmatrix}, \text{ where } \theta_{t-1} = \left(\theta_{f,t-1}, \theta_{p,t-1}\right) \text{ and} \\ \mathbf{M}\left(\theta_{t-1}\right) &\equiv \begin{pmatrix} \frac{\lambda(1-\alpha)\left(\ell_f e^{-\phi\theta_{f,t-1}} + \ell_p e^{-\phi\left(\theta_{p,t-1}-\theta_{f,t-1}\right)}\right)}{\left(\ell_f e^{-\phi\theta_{f,t-1}} - \ell_p e^{-\phi\left(\theta_{p,t-1}-\theta_{f,t-1}\right)}\right)^2} & -\frac{\lambda(1-\alpha)\ell_p e^{-\phi\left(\theta_{p,t-1}-\theta_{f,t-1}\right)}}{\left(\ell_f e^{-\phi\theta_{f,t-1}} - \ell_p e^{-\phi\left(\theta_{p,t-1}-\theta_{f,t-1}\right)}\right)^2} \\ -\frac{\lambda\alpha}{\ell_p e^{-\phi\left(\theta_{p,t-1}-\theta_{f,t-1}\right)}} & \frac{\lambda\alpha}{\ell_p e^{-\phi\left(\theta_{p,t-1}-\theta_{f,t-1}\right)}} \end{pmatrix}. \end{aligned}$$

Next, we evaluate the matrix $\mathbf{M}$ at the end points to obtain

$$\begin{aligned} \mathbf{M}\left(\mathbf{0}\right) &= \begin{pmatrix} \frac{\lambda(1-\alpha)\left(\ell_f+\ell_p\right)}{\left(\ell_f-\ell_p\right)^2} & -\frac{\lambda(1-\alpha)\ell_p}{\left(\ell_f-\ell_p\right)^2} \\ -\frac{\lambda\alpha}{\ell_p} & \frac{\lambda\alpha}{\ell_p} \end{pmatrix}, \\ \mathbf{M}\left(\mathbf{1}\right) &= \begin{pmatrix} \frac{\lambda(1-\alpha)\left(\ell_f e^{-\phi}+\ell_p\right)}{\left(\ell_f e^{-\phi}-\ell_p\right)^2} & -\frac{\lambda(1-\alpha)\ell_p}{\left(\ell_f e^{-\phi}-\ell_p\right)^2} \\ -\frac{\lambda\alpha}{\ell_p} & \frac{\lambda\alpha}{\ell_p} \end{pmatrix}. \end{aligned}$$

Recall that the dynamics is initiated at $(\theta_{f,0}, \theta_{p,0})$, which lies in the $\mathbb{FPN}$-

[45]Such a perturbation method usually gives a rigorous justification of the standard linearization method in dynamic analysis.

Equilibrium zone with $0 < \theta_{f,0} < \theta_{p,0} < 1$. This implies that[46]

$$\alpha\lambda < \ell_p < \alpha\ell_f. \tag{35}$$

And given the dynamic system (34), it can be verified that a sufficient condition for $(\theta_{f,t}, \theta_{p,t})$ to be in the $\mathbb{FPN}$-Equilibrium zone for all $t \in \mathbb{N}$ is

$$\alpha\lambda < \ell_p e^{-\phi}, \ell_p < \alpha\ell_f e^{-\phi}. \tag{36}$$

We conclude that if we start the dynamics at an initial point in the $\mathbb{FPN}$-equilibrium zone (so that (35) holds) and that ϕ is sufficiently small (so that (36) holds), the trajectory of the dynamic system (34) will always remain in the $\mathbb{FPN}$-equilibrium zone.

Finally, notice that the eigenvalues of $\phi\mathbf{M}(\mathbf{0})$ and $\phi\mathbf{M}(\mathbf{1})$ can always be made to be all less than one in absolute value as long as ϕ is sufficiently small.[47] We hence can define a cutoff $\hat{\phi} > 0$ so that (1) the eigenvalues of $\phi\mathbf{M}(\mathbf{0})$ and $\phi\mathbf{M}(\mathbf{1})$ are all less than one in absolute value, and (2) $\alpha\lambda < \ell_p e^{-\phi}$ and $\ell_p < \alpha\ell_f e^{-\phi}$, for all $\phi \in \left[0, \hat{\phi}\right)$.

Our above discussion leads to the following for the dynamic system (34):

Proposition 12 *Let $n = 1$. For each initial condition $\theta_0 = (\theta_{f,0}, \theta_{p,0})$ in the interior of the $\mathbb{FPN}$-Equilibrium zone, there exists a $\hat{\phi} > 0$ such that for all $\phi \in \left(0, \hat{\phi}\right)$, there is an open neighborhood $N_\varepsilon(\theta_{f,0}, \theta_{p,0})$ of θ_0 with a sub-neighborhood $N_\delta(\theta_{f,0}, \theta_{p,0})$ of θ_0, i.e., $N_\delta(\theta_{f,0}, \theta_{p,0}) \subseteq N_\varepsilon(\theta_{f,0}, \theta_{p,0})$, such that the dynamics (34) converges to a locally stable equilibrium (θ_f^*, θ_p^*) inside $N_\varepsilon(\theta_{f,0}, \theta_{p,0})$.*

Our explicit dynamic analysis above demonstrates that as long as ϕ is sufficiently small, i.e., the language learning speed is slow enough, then language learning dynamics initiated in the (interior) $\mathbb{FPN}$-equilibrium zone will remain in the zone and will also converge to an $\mathbb{FPN}$ equilibrium in the zone.

9.5 Omitted Proofs

Proof of Lemma 1. Let $\mathcal{F}, \mathcal{P}, \mathcal{N}$ be the measures of measurable (possibly empty) sets of agents in each minority group that choose F, P and N respectively in σ^*.[48]

[46] We can alternatively think of $(\theta_{f,0}, \theta_{p,0})$ as calculated from (34) by setting $\phi = 0$.

[47] The restriction on ϕ for the absolute-value eigenvalues to be all less than one is important. Consider a numeric setting where $\lambda = 2, \alpha = 0.6, \ell_p = 4.2, \ell_f = 8$. One can calculate that $\mathbf{M}(\mathbf{1}) = \begin{pmatrix} 17.023 & -7.898 \\ -0.28571 & 0.28571 \end{pmatrix}$, which has eigenvalues $\mu_1 = 34.314, \mu_2 = 0.30391$.

[48] Here the measurable structure on each set of agents is its collection of Borel subsets.

Convexity: Suppose $\sigma_i^*(\theta) = \sigma_i^*(\theta') = F$, i.e., for $\tilde{\theta} \in \{\theta, \theta'\}$, F is a best response, i.e., $u\left(F, \mathcal{F}, \mathcal{P}; \tilde{\theta}\right) \geq \max\left\{u\left(P, \mathcal{F}, \mathcal{P}; \tilde{\theta}\right), u\left(N, \mathcal{F}, \mathcal{P}; \tilde{\theta}\right)\right\}$, or

$$\ell_f \tilde{\theta} \leq \lambda + (n-1)\mathcal{F} + \alpha(n-1)\mathcal{P} \text{ and} \tag{37}$$
$$(\ell_f - \ell_p)\tilde{\theta} \leq (1-\alpha)\left[\lambda + (n-1)\mathcal{F} + \alpha(n-1)\mathcal{P}\right] \tag{38}$$

Conditions (37) and (38) then imply that $\forall \delta \in (0,1)$,

$$\begin{aligned} \ell_f(\delta\theta + (1-\delta)\theta') &\leq \lambda + (n-1)\mathcal{F} + \alpha(n-1)\mathcal{P} \text{ and} \\ (\ell_f - \ell_p)(\delta\theta + (1-\delta)\theta') &\leq (1-\alpha)\left[\lambda + (n-1)\mathcal{F} + \alpha(n-1)\mathcal{P}\right] \end{aligned}$$

Hence, F is also a best response for type "$\delta\theta + (1-\delta)\theta'$." The analysis for strategies P and N is analogous.

Monotonicity: We only consider the case where $\sigma_i^*(\theta) = F$and $\sigma_i^*(\theta') = P$, and the remaining cases are analogous. Recall that $\mathcal{F}, \mathcal{P}, \mathcal{N}$ are connected intervals. Suppose instead $\theta > \theta'$, and $\sigma_i^*(\theta') = P$, $\sigma_i^*(\theta) = F$. Then

$$\begin{aligned} \theta \text{ prefers } F \text{ to } P\text{, or } (\ell_f - \ell_p)\theta &\leq (1-\alpha)\left[\lambda + (n-1)\mathcal{F} + \alpha(n-1)\mathcal{P}\right] \\ \theta' \text{ prefers } P \text{ to } F\text{, or } (\ell_f - \ell_p)\theta' &\geq (1-\alpha)\left[\lambda + (n-1)\mathcal{F} + \alpha(n-1)\mathcal{P}\right] \end{aligned}$$

which contradicts $\theta > \theta'$. We hence have $\theta' \geq \theta$.

Positivity: First, F is a dominant strategy for type 0. Now consider type $\varepsilon \in \left[0, \min\left\{\frac{\lambda}{\ell_f}, \frac{(1-\alpha)\lambda}{\ell_f - \ell_p}\right\}\right]$. Given $\lambda > 1, \alpha > 0$ and $\ell_f > \ell_p$, we have

$$\begin{aligned} \ell_f \varepsilon &\leq \lambda + (n-1)\mathcal{F} + \alpha(n-1)\mathcal{P} \\ (\ell_f - \ell_p)\varepsilon &\leq (1-\alpha)\left[\lambda + (n-1)\mathcal{F} + \alpha(n-1)\mathcal{P}\right] \end{aligned}$$

regardless of $\mathcal{F}$ and $\mathcal{P}$. Hence, F is also a dominant strategy for type ε. Convexity then implies that all types in $[0, \varepsilon]$ choose F in a symmetric equilibrium. ■

Proof of Proposition 2. First, for equilibrium $\sigma^{\mathbb{FN}}$, the analysis is exactly identical to that for the binary language acquisition setting (Proposition 1) and we similarly have insufficient language acquisition.

Now consider equilibrium $\sigma^{\mathbb{FP}}$. We similarly write down the social welfare function when minority agents are either partially or fully learning the majority language with cutoff θ as:

$$W^{\mathbb{FP}}(\theta) = n\left[\begin{array}{c} 2\lambda H(\theta) + (n-1)(H(\theta))^2 + 2\alpha\lambda(1-H(\theta)) + (n-1)\alpha^2[1-H(\theta)]^2 \\ +2\alpha(n-1)[1-H(\theta)]H(\theta) - \ell_f\int_0^\theta t\,dH(t) - \ell_p\int_\theta^1 t\,dH(t) \end{array}\right]$$

Differentiate $W^{\mathbb{FP}}(\theta)$ and evaluate the derivative at $\theta = \theta_f$ in (9) to obtain

$$\begin{aligned}\left.\frac{dW^{\mathbb{FP}}(\theta)}{d\theta}\right|_{\theta=\theta_f} &\propto \left.\left[\begin{array}{c} 2(1-\alpha)\lambda + 2(n-1)(1-\alpha)H(\theta) \\ +2\alpha(1-\alpha)(n-1)(1-H(\theta)) - \ell_f\theta + \ell_p\theta \end{array}\right]\right|_{\theta=\theta_f} \\ &\propto \left[(1-\alpha)\lambda + (n-1)(1-\alpha)H(\theta_f)\alpha(1-\alpha)(n-1)(1-H(\theta_f))\right] \\ &> 0\end{aligned}$$

which implies that increasing θ_f strictly increases social welfare. Hence, there is insufficient learning in equilibrium $\sigma^{\mathbb{FP}}$.

Finally, consider equilibrium $\sigma^{\mathbb{FPN}}$. Given an arbitrary pair of cutoffs (θ_F, θ_P) with $0 < \theta_F < \theta_P < 1$, the total social welfare in such a learning outcome can be written as

$$W^{\mathbb{FPN}}(\theta_F, \theta_P) = n\left[\begin{array}{c} 2\lambda H(\theta_F) + (n-1)(H(\theta_F))^2 + (n-1)\alpha^2(H(\theta_P) - H(\theta_F))^2 \\ +2\alpha\lambda(H(\theta_P) - H(\theta_F)) + 2\alpha(n-1)(H(\theta_P) - H(\theta_F))H(\theta_F) \\ -\ell_f\int_0^{\theta_F} tdH(t) - \ell_p\int_{\theta_F}^{\theta_P} tdH(t) \end{array}\right]$$

We similarly obtain two first-order partial derivatives, evaluated as the equilibrium cutoff (θ_f, θ_p) defined in (5) and (6), as

$$\begin{aligned}\left.\frac{\partial W^{\mathbb{FPN}}(\theta_F, \theta_P)}{\partial\theta_F}\right|_{(\theta_f,\theta_p)} &\propto \left[\begin{array}{c} (1-\alpha)\lambda + (1-\alpha)(n-1)H(\theta_f) \\ +\alpha(1-\alpha)(n-1)(H(\theta_p) - H(\theta_f)) \end{array}\right] \\ \left.\frac{\partial W^{\mathbb{FPN}}(\theta_F, \theta_P)}{\partial\theta_P}\right|_{(\theta_f,\theta_p)} &\propto \left[\alpha\lambda + \alpha(n-1)H(\theta_f) + \alpha^2(n-1)(H(\theta_p) - H(\theta_f))\right]\end{aligned}$$

And we immediately obtain

$$\left.\frac{\partial W^{\mathbb{FPN}}(\theta_F, \theta_P)}{\partial\theta_F}\right|_{(\theta_f,\theta_p)} > 0 \text{ and } \left.\frac{\partial W^{\mathbb{FPN}}(\theta_F, \theta_P)}{\partial\theta_P}\right|_{(\theta_f,\theta_p)} > 0,$$

or there is insufficient acquisition at both partial and full learning levels. ■

Proof of Proposition 3. Consider Part [I], i.e., $L_p \geq 1$.

First, condition (14) in the uniform setting reduces to $L_f \leq \min\{1, 1 - \alpha(1 - L_p)\}$. Hence, given $L_p \geq 1$, equilibrium $\mathbb{F}$ exists if $L_f \leq 1$.

For equilibrium $\mathbb{FN}$, i.e., condition (13), $\theta_f = \frac{\lambda}{\ell_f - (n-1)} \in (0, 1)$, is equivalent to $L_f > 1$, while $u_i\left(P; \sigma^{\mathbb{FN}}, \theta_f\right) \leq u_i\left(N; \sigma^{\mathbb{FN}}, \theta_f\right)$ reduces to

$$\frac{\lambda}{\ell_f - (n-1)} \geq \frac{\alpha\lambda}{\ell_p - \alpha(n-1)} \Leftrightarrow \frac{\ell_p}{\alpha} \geq \ell_f \Rightarrow L_p \geq L_f.$$

Hence, equilibrium $\mathbb{FN}$ arises if $1 < L_f \leq L_p$.

Equilibrium $\mathbb{FP}$ requires two conditions $\theta_f < 1$ and $u_i\left(P;\sigma^{\mathbb{FP}},1\right) \geq 1$. For the cutoff θ_f from (9), shown explicitly in (39), to be in $(0,1)$, we need $L_f > 1-\alpha\left(1-L_p\right)$. While the second condition $u_i\left(P;\sigma^{\mathbb{FP}},1\right) \geq 1$, i.e., type $\theta = 1$ prefers P to N, reduces to

$$\theta_f = \frac{(1-\alpha)\left[\lambda+\alpha\left(n-1\right)\right]}{\left(\ell_f-\ell_p\right)-(1-\alpha)^2\left(n-1\right)} \geq \frac{\ell_p-\alpha\lambda-\alpha^2\left(n-1\right)}{\alpha\left(1-\alpha\right)\left(n-1\right)} \equiv \frac{R\left(\alpha\right)}{\alpha\left(1-\alpha\right)\left(n-1\right)}. \tag{39}$$

It can be verified that $R\left(\alpha\right)/\left[\alpha\left(1-\alpha\right)\left(n-1\right)\right] < 1$ is equivalent to $L_p < 1$, contradicting $L_p \geq 1$. Hence, equilibrium $\mathbb{FP}$ cannot exist if $L_p \geq 1$.

Finally, consider equilibrium $\mathbb{FPN}$. Recall that the requirement for this equilibrium to exist is $0 < \theta_f < \theta_p < 1$, where the interior cutoffs are calculated in (15) and (16). Observe that $L_f > L_p$ (or $\ell_f > \frac{\ell_p}{\alpha}$) implies $\theta_f < \theta_p$. Moreover, since $\ell_f > \ell_p$, we have

$$\begin{aligned} L_p &\geq 1 \Leftrightarrow \alpha\lambda\left(\ell_f-\ell_p\right) \leq \left(\ell_f-\ell_p\right)\left[\ell_p-\alpha\left(n-1\right)\right], \\ \ell_f &> \frac{\ell_p}{\alpha} \Leftrightarrow \left(\ell_f-\ell_p\right)\left[\ell_p-\alpha\left(n-1\right)\right] \leq \ell_p\left(\ell_f-\ell_p\right)+\left(n-1\right)\left(2\alpha\ell_p-\alpha^2\ell_f-\ell_p\right), \end{aligned}$$

which jointly imply that $\theta_p < 1$, and the denominator of θ_f is positive, i.e., $\theta_f > 0$.

Now consider Part [II], i.e., $L_p < 1$. We will use the following key parameters:

$$\begin{aligned} \bar{\alpha} &= \frac{\sqrt{\lambda^2+4\left(n-1\right)\ell_p}-\lambda}{2\left(n-1\right)},\ \overline{L}_p = \frac{\ell_p}{\bar{\alpha}\left(\lambda+n-1\right)}, \\ G &= \frac{\ell_p^2-\alpha\lambda\ell_p-\left(n-1\right)\left(2\alpha\ell_p-\ell_p\right)}{\left[\ell_p-\alpha\lambda-\left(n-1\right)\alpha^2\right]\left(\lambda+n-1\right)}. \end{aligned} \tag{40}$$

For equilibrium $\mathbb{F}$, condition (14) $L_f \leq \min\left\{1, 1-\alpha\left(1-L_p\right)\right\}$ reduces to

$$L_f \leq 1-\alpha\left(1-L_p\right).$$

For equilibrium $\mathbb{FN}$, our discussion in Part [I] shows that this equilibrium exists if and only if $1 < L_f \leq L_p$, which cannot hold when $L_p < 1$.

Now consider equilibrium $\mathbb{FP}$. As discussed in Part [I], "$\theta_f < 1$" is equivalent to $L_f > 1-\alpha\left(1-L_p\right)$, while "$u_i\left(P;\sigma^{\mathbb{FP}},1\right) \geq 1$" is shown in (39), where $R\left(\alpha\right)/\left[\alpha\left(1-\alpha\right)\left(n-1\right)\right] < 1$, given $L_p < 1$. In addition, we verify that $R\left(\bar{\alpha}\right) = 0$ (we ignore the other negative root of $R\left(\alpha\right) = 0$). Depending on the sign of $R\left(\alpha\right)$, we have two cases: If $\alpha \geq \bar{\alpha}$ or $L_p \leq \overline{L}_p = \frac{\ell_p}{\bar{\alpha}(\lambda+n-1)}$, we have $R\left(\alpha\right) \leq 0$ and hence (39) is automatic. Notice that $L_p \leq \overline{L}_p$ is always true if $\overline{L}_p \geq 1$ since we have assumed $L_p < 1$. If $\overline{L}_p < 1$ and $L_p \in \left(\overline{L}_p, 1\right)$, we have $R(\alpha) > 0$ and (39) reduces to

$$\ell_f \leq \frac{\alpha\lambda\ell_p - \ell_p^2 + (n-1)(2\alpha\ell_p - \ell_p)}{\alpha\lambda - \ell_p + (n-1)\alpha^2} \Longleftrightarrow L_f \leq G. \tag{41}$$

Since $G > 1 - \alpha(1 - L_p)$, $1 - \alpha(1 - L_p) < L_f \leq G$ is hence well defined.

Finally, consider equilibrium $\mathbb{FPN}$. As before, $\theta_p > \theta_f$ is equivalent to $\ell_f > \frac{\ell_p}{\alpha}$, or $L_f > L_p$. We further need $\theta_f > 0$ and $\theta_p < 1$. Now rewrite $\theta_p < 1$ to be

$$\begin{aligned} \alpha\lambda(\ell_f - \ell_p) &< \ell_p(\ell_f - \ell_p) + (n-1)\left(2\alpha\ell_p - \alpha^2\ell_f - \ell_p\right) \text{ or} \\ \left(\ell_p - \alpha\lambda - \alpha^2(n-1)\right)\ell_f &= R(\alpha)\ell_f > \ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p \end{aligned} \tag{42}$$

Observe that (42) is similar to (41) for equilibrium $\mathbb{FP}$, since the two equilibria are similar. However, the ranges of α are different across the two equilibria. Recall that $R(\alpha) = \ell_p - \alpha\lambda - \alpha^2(n-1)$, with $R(\bar{\alpha}) = 0$, $R(\alpha) > 0$ for $\alpha < \bar{\alpha}$ and $R(\alpha) < 0$ for $\alpha > \bar{\alpha}$. We consider two familiar cases.

Case 1. $\overline{L}_p \geq 1$. Such an $\overline{L}_p$, together with $1 > L_p$, is equivalent to $\alpha > \frac{\ell_p}{\lambda+n-1} \geq \bar{\alpha}$, implying that $R(\alpha) < 0$. Hence condition (42) can be rewritten as

$$\ell_f < \frac{\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p}{R(\alpha)} = \frac{\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p}{\ell_p - \alpha\lambda - \alpha^2(n-1)}. \tag{43}$$

If the numerator of the RHS of (43) is non-negative, i.e., if $\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p \geq 0$, then expression (43) can never hold since the RHS of (43) is non-positive and $\ell_f > 0$. While if the numerator is negative, which happens when $\alpha > \frac{\ell_p+n-1}{\lambda+2(n-1)}$, then

$$\frac{\ell_p}{\alpha} < \ell_f < \frac{\alpha\lambda\ell_p - \ell_p^2 - (1-2\alpha)(n-1)\ell_p}{\alpha\lambda - \ell_p + \alpha^2(n-1)}. \tag{44}$$

However, it can be verified that the above range for ℓ_f is empty given $\alpha > \frac{\ell_p}{\lambda+n-1}$. This discussion implies that equilibrium $\mathbb{FPN}$ does not exist if $\alpha > \frac{\ell_p}{\lambda+n-1} \geq \bar{\alpha}$ or equivalently if $\overline{L}_p \geq 1 > L_p$.

Case 2. $\overline{L}_p < 1$, or equivalently $\frac{\ell_p}{\lambda+n-1} < \bar{\alpha}$. We either have $L_p \in \left(\overline{L}_p, 1\right)$, i.e., $\alpha \in \left(\frac{\ell_p}{\lambda+n-1}, \bar{\alpha}\right)$, or $L_p \in \left(\frac{\ell_p}{\lambda+n-1}, \overline{L}_p\right]$, i.e., $\alpha \in [\bar{\alpha}, 1)$. For $\alpha \in \left(\frac{\ell_p}{\lambda+n-1}, \bar{\alpha}\right)$, $R(\alpha) > 0$, similar to our discussion for equilibrium $\mathbb{FP}$. Hence, condition (42) is equivalent to[49]

$$\ell_f > \frac{\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p}{\ell_p - \alpha\lambda - \alpha^2(n-1)} \Longleftrightarrow L_f > G,$$

the opposite of that for equilibrium $\mathbb{FP}$. Next, for $\alpha \in [\bar{\alpha}, 1)$, an argument similar to (43) and (44) implies that equilibrium $\mathbb{FPN}$ cannot exist here.[50]

Summarizing, if $L_p < 1$, equilibrium $\mathbb{FPN}$ exists whenever we have $L_p \in \left(\overline{L}_p, 1\right)$ and

$$\ell_f > \frac{\ell_p}{\alpha} \text{ and } \ell_f > \frac{\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p}{\ell_p - \alpha\lambda - \alpha^2(n-1)}.$$

Since

$$L_p < 1 \Longrightarrow \frac{\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p}{\ell_p - \alpha\lambda - \alpha^2(n-1)} > \frac{\ell_p}{\alpha},$$

we conclude that equilibrium $\mathbb{FPN}$ arises when $L_p \in \left(\overline{L}_p, 1\right)$ and $L_f > G$.

Finally, uniqueness of equilibrium is immediate from the observation that the parameter constellations for the four equilibria for each case of $L_p \geq 1$ and $L_p < 1$ form a partition (exhaustive and mutually exclusive) of the entire parameter space. ■

Proof of Proposition 6. For the binary setting, we can calculate that

$$W^B\left(\widehat{\theta}_f^{\varepsilon}; \varepsilon\right) = \frac{2\lambda^2}{(1-\varepsilon)\ell_f} - \frac{\lambda^2}{2(1-\varepsilon)^2\ell_f}$$

which enables us to directly obtain

$$\left.\frac{dW^B\left(\widehat{\theta}_f^{\varepsilon}; \varepsilon\right)}{d\varepsilon}\right|_{\varepsilon=0} = \left.\frac{(1-2\varepsilon)\lambda^2}{\ell_f(1-\varepsilon)^3}\right|_{\varepsilon=0} = \frac{\lambda^2}{\ell_f} > 0.$$

[49] Notice that the numerator of G is positive, i.e., $\ell_p^2 - \alpha\lambda\ell_p + (1-2\alpha)(n-1)\ell_p > 0$, equivalently $\alpha < \frac{\ell_p+n-1}{\lambda+2(n-1)}$, given that $\alpha < \frac{\ell_p}{\lambda+n-1}$ $(L_p < 1)$.

[50] In our proof, we have not discussed the possibility where $\bar{\alpha} > 1$. This case is irrelevant and our arguments associated with such $\bar{\alpha}$ are then vacuously true.

In our setting with partial learning, we have

$$\begin{aligned} W^{\mathbb{FPN}}\left(\theta_f^{\varepsilon},\theta_p^{\varepsilon};\varepsilon\right) &= 2\lambda\theta_f^{\varepsilon}+2\alpha\lambda\left(\theta_p^{\varepsilon}-\theta_f^{\varepsilon}\right)-(1-\varepsilon)\ell_f\frac{(\theta_f^{\varepsilon})^2}{2}-\ell_p\frac{(\theta_p^{\varepsilon})^2}{2} \\ &= \frac{\lambda^2\left\{\begin{array}{c} -4\ell_p^2+3\alpha^2(1-\varepsilon)^2\ell_f^2-\alpha^2\ell_p^2+3(1-\varepsilon)\ell_f\ell_p+ \\ 8\alpha\ell_p^2-3\alpha^2(1-\varepsilon)\ell_f\ell_p-6\alpha(1-\varepsilon)\ell_f\ell_p \end{array}\right\}}{2\ell_p\left((1-\varepsilon)\ell_f-\ell_p\right)^2}. \end{aligned}$$

We further obtain

$$\begin{aligned} \left.\frac{dW^{\mathbb{FPN}}\left(\theta_f^{\varepsilon},\theta_p^{\varepsilon};\varepsilon\right)}{d\varepsilon}\right|_{\varepsilon=0} &= \lambda^2\ell_f(1-\alpha)^2\left.\frac{5\ell_p-3\ell_f+3\varepsilon\ell_f}{2\left(\ell_p-\ell_f+\varepsilon\ell_f\right)^3}\right|_{\varepsilon=0} \\ &= \frac{\lambda^2\ell_f(1-\alpha)^2\left(3\ell_f-5\ell_p\right)}{2\left(\ell_f-\ell_p\right)^3}. \end{aligned}$$

Hence, we have

$$\left.\frac{dW^{\mathbb{FPN}}\left(\theta_f^{\varepsilon},\theta_p^{\varepsilon};\varepsilon\right)}{d\varepsilon}\right|_{\varepsilon=0} < 0$$

if $\ell_f \in \left(\ell_p, \frac{5}{3}\ell_p\right)$. ■

Proof of Proposition 7. We establish the result by directly comparing the majority agents' total welfare before and after banning partial learning in the language economy.

First, observe that if currently the language equilibrium is either an $\mathbb{FP}$ or $\mathbb{FPN}$ equilibrium, banning partial learning will lead to an $\mathbb{FN}$ equilibrium with an *interior* cutoff

$$\theta_f^{FN} = \frac{\lambda}{\ell_f-(n-1)} \in (0,1).$$

And majority agents' total welfare after banning partial learning is:

$$W_M^B(\alpha) = \lambda n\theta_f^{FN} = \frac{n\lambda^2}{\ell_f-(n-1)}.$$

Now suppose the equilibrium before the ban is either an $\mathbb{FPN}$ equilibrium, with cutoffs θ_f^{FPN} and θ_p^{FPN} in (15) and (16), or an $\mathbb{FP}$ equilibrium, with cutoff $\theta_f^{\mathbb{FP}}$ in (9) or (39). The majority agents' welfare without the ban can be written as, respectively:

$$\begin{aligned} W_M^{\mathbb{FPN}}(\alpha) &= \lambda n[\theta_f^{FPN} + \alpha\left(\theta_p^{FPN} - \theta_f^{FPN}\right)] = \frac{\lambda^2 n[\ell_p(1-\alpha)^2 + \alpha^2(\ell_f - \ell_p)]}{\ell_p(\ell_f - \ell_p) + (n-1)(2\alpha\ell_p - \alpha^2\ell_f - \ell_p)}, \\ W_M^{\mathbb{FP}}(\alpha) &= \lambda n[\theta_f^{FP} + \alpha\left(1 - \theta_f^{FP}\right)] = \lambda n\left\{\frac{(1-\alpha)^2[\lambda + \alpha(n-1)]}{(\ell_f - \ell_p) - (1-\alpha)^2(n-1)} + \alpha\right\}. \end{aligned}$$

We first compare $W_M^B(\alpha)$ with $W_M^{\mathbb{FPN}}(\alpha)$. Note that $\theta_f^{FN} = \theta_p^{FPN} = \theta_f^{FPN}$ if $\alpha = \ell_p/\ell_f$, which implies that $W_M^{\mathbb{FPN}}(\alpha) = W_M^B(\alpha)$ if $\alpha = \ell_p/\ell_f$.[51] We can further calculate that

$$\frac{dW_M^{\mathbb{FPN}}(\alpha)}{d\alpha} = 0 \text{ if } \alpha = \ell_p/\ell_f \text{ and } \frac{d^2W_M^{\mathbb{FPN}}(\alpha)}{d\alpha^2} > 0.$$

Hence the welfare function $W_M^{\mathbb{FPN}}(\alpha)$ achieves its global minimum at $\alpha = \ell_p/\ell_f$, which implies that $W_M^{\mathbb{FPN}}(\alpha) > W_M^B(\alpha)$ for all $\alpha > \ell_p/\ell_f$.

Now we compare $W_M^B(\alpha)$ and $W_M^{\mathbb{FP}}(\alpha)$. First, given the $\mathbb{FP}$ equilibrium, it is sufficient to consider $\alpha > \ell_p/\ell_f$. Since $W_M^{\mathbb{FP}}(\alpha)$ is strictly increasing in ℓ_p and $\ell_p < \alpha\ell_f$, we have

$$W_M^{\mathbb{FP}}(\alpha) > W_M^{\mathbb{FP}}(\alpha)\big|_{\ell_p = \alpha\ell_f} = \hat{W}_M^{\mathbb{FP}}(\alpha) = \lambda n\left[\frac{(1-\alpha)(\lambda + \alpha(n-1))}{\ell_f - (1-\alpha)(n-1)} + \alpha\right].$$

We can also verify that[52]

$$\frac{d\hat{W}_M^{\mathbb{FP}}(\alpha)}{d\alpha} = \frac{\lambda n\ell_f(\ell_f - \lambda - (n-1))}{(\ell_f - \alpha - n + n\alpha + 1)^2} + \lambda n > 0 \text{ and } \hat{W}_M^{\mathbb{FP}}(0) = W_M^B(\alpha).$$

Hence, we have $W_M^{\mathbb{FP}}(\alpha) > W_M^B(\alpha)$ for all $\alpha > \ell_p/\ell_f$ as well. ■

Proof of Proposition 10. First consider game G^B. By Cramer's rule, for $1 \le i \le K$, we have

$$\theta_f^i = \frac{\det(D_i)}{\det(\mathbf{A})}$$

where D_i is the matrix that replaces the i^{th} column of $\mathbf{A}$ by π_0

[51] Recall that $\alpha = \ell_p/\ell_f$ marks the cutoff between $\mathbb{FPN}$ equilibrium and $\mathbb{FN}$ equilibrium in the parameter space (see Figure 3). And everything else fixed, as α increases from ℓ_p/ℓ_f to 1, we first enter the $\mathbb{FPN}$ equilibrium zone and then the $\mathbb{FP}$ equilibrium zone.

[52] Recall that since $0 < \theta_f^{FN} < 1$, we have $\ell_f > \lambda + n - 1$.

$$D_i = \begin{bmatrix} \ell_f & -\pi_2 & \cdots & \pi_0 & \cdots & -\pi_K \\ -\pi_1 & \ell_f & \cdots & \pi_0 & \cdots & -\pi_K \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\pi_1 & -\pi_2 & \cdots & \pi_0 & \cdots & \ell_f \end{bmatrix}$$

Since $\det(\mathbf{A}) > 0$, we only need to show that if ℓ_f is sufficiently large, then $\det(D_i) < \det(D_{i+1})$ for all $1 \leq i \leq K-1$. Given the special form of D_i, it is easiest to calculate $\det(D_i)$ directly. To do that, notice that the j^{th} row differs from the i^{th} row only at the j^{th} component ($j \neq i$). Subtract each row by the i^{th} row to obtain:

$$\det(D_i) = \det \begin{bmatrix} \ell_f + \pi_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \ell_f + \pi_2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -\pi_1 & -\pi_2 & \cdots & \pi_0 & \cdots & -\pi_K \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \ell_f + \pi_K \end{bmatrix}$$

Since the i^{th} column only has a non-zero i^{th} component, a direct calculation leads to

$$\det(D_i) = \pi_0 (\ell_f + \pi_1) \cdots (\ell_f + \pi_{i-1}) (\ell_f + \pi_{i+1}) \cdots (\ell_f + \pi_K).$$

Hence, $\det(D_i) < \det(D_{i+1})$ due to $\pi_0 > \pi_1 > \cdots > \pi_K$, which implies that

$$\theta_f^1 < \theta_f^2 < \cdots < \theta_f^K.$$

Next, we prove the monotonicity result for the game G^T. We start with some definitions. Consider the space of $m \times n$ matrices of real numbers, $M_{m\times n}(\mathbb{R})$. If $m = n$, $M_{m\times n}(\mathbb{R})$ consists of all square matrices of real numbers of size n, while if $n = 1$, $M_{m\times n}(\mathbb{R}) = \mathbb{R}^m$. For matrix $A = (a_{ij}) \in M_{m\times n}(\mathbb{R})$, define the maximal absolute value norm by $\|A\| := \sup |a_{ij}|$.

Recall that for an invertible $n \times n$ real matrix A (i.e., $A \in \mathrm{GL}_n(\mathbb{R})$) and $b \in \mathbb{R}^n$, the equation $Ax = b$ has a unique solution $x = A^{-1}b$. Here $\mathrm{GL}_n(\mathbb{R})$ denotes the general linear group over $\mathbb{R}$, which is the group of $n \times n$ invertible matrices of real numbers. As a result, we can consider the solution x as a function on A and b :

$$x = x(A, b) : \mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^n \mapsto \mathbb{R}^n, (A, b) \mapsto A^{-1}b.$$

Then x is continuous on (A, b) with respect to the topology defined by $\|\cdot\|$.

As before, define $\mathbb{A}$ in the form of a block matrix, $\mathbb{A} = \left[\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}\right]$ and recall $\mathbf{\Pi}_0 = \begin{bmatrix} \mathbf{\Pi}_0^1 \\ \mathbf{\Pi}_0^2 \end{bmatrix}$. The equilibrium condition (29), $\mathbb{A}\theta = \mathbf{\Pi}_0$, can be equivalently written as:

$$\begin{cases} (A - BD^{-1}C)\,\theta_f = \mathbf{\Pi}_0^1 - BD^{-1}\mathbf{\Pi}_0^2, \\ (D - CA^{-1}B)\,\theta_p = \mathbf{\Pi}_0^2 - CA^{-1}\mathbf{\Pi}_0^1, \end{cases} \tag{45}$$

where $\theta_f = \left(\theta_f^1, \theta_f^2, \ldots, \theta_f^K\right)$ and $\theta_p = \left(\theta_p^1, \theta_p^2, \ldots, \theta_p^K\right)$.

Next, define a new variable $\overline{\ell}_f = \ell_f - \ell_p$ and so the condition of $\ell_f > \ell_p$ is simply $\overline{\ell}_f > 0$. It is hence sufficient to just consider the two (independent) variables $\overline{\ell}_f$ and ℓ_p for the equilibrium condition $\mathbb{A}\theta = \mathbf{\Pi}_0$. It is clear from the definition of $\mathbb{A}$ that matrix A depends on $\overline{\ell}_f$ and matrix D depends on ℓ_p, and matrices B and C are constant matrices independent of $\overline{\ell}_f$ and ℓ_p. Notice that the inverse matrices A^{-1} and D^{-1} can be regarded as sufficiently small perturbations whenever $\overline{\ell}_f$ and ℓ_p are both sufficiently large—indeed the larger $\overline{\ell}_f$ (resp., ℓ_p) is, the smaller A^{-1} (resp., D^{-1}) is.

Now consider the first equation in (45). Fix a sufficiently large $\overline{\ell}_f$. By continuity of θ_f, if ℓ_p is sufficiently large, i.e., if D^{-1} is small enough, then in comparing the θ_f^i's in θ_f, we can ignore the terms with D^{-1} and the analysis is qualitatively identical to our analysis for the game G^B. Similarly, fix a sufficiently large ℓ_p. If $\overline{\ell}_f$ is also sufficiently large, i.e., A^{-1} is small enough, again the same argument for the game G^B holds for the rankings of the θ_p^i's in θ_p. We conclude that for sufficiently large $\overline{\ell}_f$ and ℓ_p, we hence have respectively

$$\theta_f^1 < \theta_f^2 < \cdots < \theta_f^K \text{ and } \theta_p^1 < \theta_p^2 < \cdots < \theta_p^K.$$

Finally, we impose the condition $\theta_f^i < \theta_p^i$ so that the language equilibrium is well defined. Consider the following matrix $\widehat{A}$, which is a general form for the diagonal matrices A and D in $\mathbb{A}$ ($\ell = \ell_f - \ell_p$ and $\widehat{\pi}_i = (1-\alpha)^2 \pi_i$ in matrix A, while $\ell = \ell_p$ and $\widehat{\pi}_i = \alpha^2 \pi_i$ in matrix D)

$$\begin{aligned} \widehat{A} &= \begin{bmatrix} \ell & -\widehat{\pi}_2 & \cdots & -\widehat{\pi}_K \\ -\widehat{\pi}_1 & \ell & \cdots & -\widehat{\pi}_K \\ \vdots & \vdots & \ddots & \vdots \\ -\widehat{\pi}_1 & -\widehat{\pi}_2 & \cdots & \ell \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} -\widehat{\pi}_1 & -\widehat{\pi}_2 & \cdots & -\widehat{\pi}_K \end{bmatrix} + \begin{bmatrix} \ell + \widehat{\pi}_1 & & & \\ & \ell + \widehat{\pi}_2 & & \\ & & \ddots & \\ & & & \ell + \widehat{\pi}_K \end{bmatrix} \end{aligned}$$

where the last matrix is a diagonal matrix.

By the Sherman-Morrison formula, we have

$$\det(\widehat{A}) = \left(1 - \begin{bmatrix} -\widehat{\pi}_1 & -\widehat{\pi}_2 & \cdots & -\widehat{\pi}_K \end{bmatrix} \begin{bmatrix} \ell + \widehat{\pi}_1 & & & \\ & \ell + \widehat{\pi}_2 & & \\ & & \ddots & \\ & & & \ell + \widehat{\pi}_K \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right) \cdot \det \begin{bmatrix} \ell + \widehat{\pi}_1 & & & \\ & \ell + \widehat{\pi}_2 & & \\ & & \ddots & \\ & & & \ell + \widehat{\pi}_K \end{bmatrix}$$

which leads to

$$\begin{aligned} \det(\widehat{A}) &= \prod_{i=1}^{K} (\ell + \widehat{\pi}_i) - \sum_{i=1}^{K} \widehat{\pi}_i \prod_{j \neq i} (\ell + \widehat{\pi}_j) \\ &= \ell^K - \sum_{i \neq j}^{K} (\widehat{\pi}_i \widehat{\pi}_j)\, \ell^{K-2} + \cdots \end{aligned}$$

where we have expanded the expression of $\det(\widehat{A})$, focusing on the two **leading** terms in the expansion. Given that $\ell \gg 0$, the above expression of $\det(\widehat{A})$ implies that a larger $\widehat{\pi}_i$ is associated with a smaller $\det(\widehat{A})$. Recall that $\widehat{\pi}_i = (1-\alpha)^2 \pi_i$ in matrix A and $\widehat{\pi}_i = \alpha^2 \pi_i$ in matrix D. Therefore, by imposing $(1-\alpha)^2 < \alpha^2$, i.e., α is large, we can guarantee that for all i, $0 < \theta_f^i < \theta_p^i < 1$, together with sufficiently large ℓ_f and ℓ_p.[53] ■

[53] Notice that in the process of our reasoning, we have always focused on leading terms and omitting insignificant terms. As such, the condition $(1-\alpha)^2 < \alpha^2$, though intuitive, should not be considered as a "precise" sufficient condition for $\theta_f^i < \theta_p^i$.